COMPUTATIONAL COMPLEXITY OF THEORIES OF RESIDUATED STRUCTURES

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The present talk is based on joint work with C. J. Van Alten [5, 8].

Residuated structures are used as algebraic models of a wide variety of propositional logics; they play particularly prominent role in the study of substructural logics. Equational and atomic theories of such structures correspond to sequents provable in logics; quasi-equational and Horn theories correspond to statements of derivability of sequents from sets of sequents; universal theories correspond to inferences of Boolean combinations of sequents from Boolean combinations of sequents.

In this talk, we are concerned with the following types of residuated structures. A *residuated ordered groupoid* (*rog*, for short) is a tuple $\langle A, \leqslant, \circ, \backslash, /, \rangle$, where $\langle A, \leqslant \rangle$ is a partially ordered set, and $\circ$, $\backslash$, and $/$ are binary operations on A satisfying the residuation condition:

$$a \circ b \leqslant c \iff b \leqslant a \backslash c \iff a \leqslant c/b. \tag{1}$$

The class of all rogs shall be denoted by $\mathcal{ROG}$. A *residuated distributive lattice-ordered groupoid* (*rdg*, for short) is a tuple $\langle A, \vee, \wedge, \circ, \backslash, / \rangle$, where $\langle A, \vee, \wedge \leqslant \rangle$ is a distributive lattice, and $\circ$, $\backslash$, and $/$ are binary operations on A satisfying (1). The class of all rdgs shall be denoted by $\mathcal{RDG}$.

Theories of rogs shall be stated in a first-order language in the signature with operation symbols $\circ$, $\backslash$, and $/$, and the relational symbol $\leqslant$. Theories of rdgs shall be stated in a first-order language in the signature with operation symbols $\vee$, $\wedge$, $\circ$, $\backslash$, $/$, and the relational symbol $=$ (the relational symbol $\leqslant$ is definable in the standard way: $a \leqslant b := a \wedge b = a$). For both rogs and rdgs, terms and valuations are defined in the standard way. The evaluation of formulas and validity are defined as in the standard model theory.

The *atomic theory* of $\mathcal{ROG}$ is the set of the atomic formulas (i.e., expressions of the form $s \leqslant t$) valid in $\mathcal{ROG}$. The *Horn theory* of $\mathcal{ROG}$ is the set of formulas of the form $\alpha_1 \dot{\wedge} \dots \dot{\wedge} \alpha_n \Rightarrow \alpha$, where $\alpha_1, \dots, \alpha_n$ and α are all atomic, valid in $\mathcal{ROG}$. The *universal theory* of $\mathcal{ROG}$ is the set of formulas $\forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n)$, where φ is a Boolean combination of atomic formulas, valid in $\mathcal{ROG}$. The *equational theory* of $\mathcal{RDG}$ is the set of equations valid in $\mathcal{RDG}$. The *quasi-equational theory* of $\mathcal{RDG}$ is the set of quasi-equations (i.e., expressions of the form $\alpha_1 \dot{\wedge} \dots \dot{\wedge} \alpha_n \Rightarrow \alpha$, where $\alpha_1, \dots, \alpha_n$ and α are all equations) valid in $\mathcal{RDG}$. The *universal theory* of $\mathcal{RDG}$ is the set of formulas $\forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n)$, where φ is a Boolean combination of equations, valid in $\mathcal{RDG}$. It was established by Aarts and Trautwein [1] that the atomic theory of $\mathcal{ROG}$, and by Buszkowski [2] that the Horn theory of $\mathcal{ROG}$, are polynomial-time decidable. We present here results on the complexity of universal theory of $\mathcal{ROG}$, as well as universal and quasi-equational theories of $\mathcal{RDG}$:

Theorem 1. *The universal theory of $\mathcal{ROG}$ is coNP-complete.*

Theorem 2. *The universal and quasi-equational theories of $\mathcal{RDG}$ are both EXPTIME-complete.*

The upper bounds are obtained through the use of partial algebras [9, 6, 7]. We show that, if a quantifier-free formula φ is satisfiable in a rog, this is witnessed by a partial rog of size polynomial in the size of φ; moreover, we identify a set of polynomial-time verifiable structural conditions ensuring that a partial structure is a partial rog. This gives us a non-deterministic polynomial-time algorithm for checking satisfiability of quantifier-free formulas in $\mathcal{ROG}$: we guess a partial structure of size polynomial in the size of the input formula φ, check that this structure is a partial rog, and lastly check if φ is satisfied in the structure we guessed. Analogously, we show that, if a quantifier-free formula φ is satisfiable in a rdg, this is witnessed by a partial rdg of size exponential in the size of φ; moreover, there exists a deterministic algorithm for checking if such a partial rdg exists for a formula (the algorithm is based on a structural characterization of partial rdgs using exponential-time verifiable properties of partial structures). In the structural characterization of partial rogs and partial rdgs we essentially rely on the relational frame theory developed by Dunn [3, 4].

The lower bound for Theorem 1 follows from a simple observation that the universal theory of a class of non-trivial structures is as computationally hard the Boolean logic. The lower bound for Theorem 2 is obtained by reduction from a corridor tiling game through satisfiability in a modal logic with the universal modality.

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