CONSTRUCTIVE INTERPRETATIONS OF LOGICAL AND LOGICAL-MATHEMATICAL LANGUAGES

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An survey of the author's main results obtained during his lifetime is offered in the context of research in the field of constructive logic conducted in the USSR and Russia (and not only) in the last 50 years.

1. Propositional logic of recursive realizability. By constructive semantics we will understand intuitionistic interpretations of logical and logical-mathematical languages, in which the concept of an effective operation is explained through the concept of an algorithm. The first semantics of this kind called recursive realizability was proposed by S. C. Kleene for the formal arithmetic language. On its basis, various variants of the concept of a realizable predicate (in particular, propositional) formula are possible, of which irrefutability and effective realizability deserve the most attention. A predicate formula is called irrefutable if the universal closure of any arithmetic example of this formula is realizable. A closed predicate formula is called effectively realizable if there exists an algorithm that allows for any closed arithmetic instance of this formula to find its realization. The author proved [1] that in general these concepts do not coincide, but for propositional formulas the question of their coincidence or difference remains open.

In 1932 A. N. Kolmogorov proposed an informal interpretation of intuitionistic logic as the logic of problems. 30 years later, this idea was somewhat refined by Medvedev in the form of the concept of finite validity for propositional formulas. In 1963, Medvedev [2] published an erroneous proof of the theorem that every realizable propositional formula is finitely valid. The author has constructed a counterexample to this statement. In this case, the finite model property of the Medvedev logic and Yankov's characteristic formulas [3] were significantly used. A detailed exposition of this result is available in [4]. Medvedev proved the completeness of fragments of the intuitionistic propositional calculus IPC without negation and disjunction with respect to finite validity. From the above-mentioned erroneous theorem, a similar result was obtained for the propositional logic of recursive realizability. The author proved that the completeness of these fragments of the calculus IPC with respect to recursive realizability really holds. A detailed proof is available in [5].

2. Predicate logic of recursive realizability. The propositional logic of recursive realizability is difficult to investigate. The situation is much better with the predicate logic of recursive realizability. The author proved [6] that the set of all realizable predicate formulas is not arithmetic. The proof was based on the Tennenbaum theorem that there are no recursive non-standard models of arithmetic. The technique developed at the same time allowed solving many issues related to the predicate logic of recursive realizability. The scheme theorem is technically important. The concept of a scheme over the arithmetic language was introduced by M. M. Kipnis [7]. A scheme is a formula in the mixed language of arithmetic and predicate logic. For schemes, all concepts of realizability are introduced by analogy with predicate formulas. The scheme theorem [1] states that for any scheme it is possible to efficiently construct a predicate formula that is realizable in one sense or another if and only if the original scheme is realizable in the same sense. This made it possible to show the difference between the mentioned variants of realizability for predicate formulas, as well as their difference from the concept of uniform realizability, meaning the existence of a single realization for all closed arithmetic instances of a closed predicate formula.

It follows from the Nelson theorem that the intuitionistic predicate calculus IQC is sound with respect to recursive realizability. It was shown by Rose that the calculus IPC is not complete with respect to this semantics. Subsequently, Markov formulated a logical principle, now called the Markov principle, meaning, in particular, the realizability of the predicate formula

$$\forall x\,(P(x) \vee \neg P(x)) \to (\neg\neg\exists x\, P(x) \to \exists x\, P(x))$$

(denote it M), non-deducible in IQC. A scheme ECT over the arithmetic language, called the extended Church thesis, is sound with respect to the semantics of recursive realizability. In the presence of Markov's principle, this scheme is equivalent to the scheme

$$\forall x\,(\neg A(x) \to \exists y\, B(x,y)) \to \exists z\, \forall x\,(\neg A(x) \to \exists y\,(\{z\}(x) = y \,\&\, B(x,y))),$$

where $\{z\}$ denotes a partial recursive function with a Gödel number z. The scheme theorem allows us to replace this scheme with a predicate formula; we denote it ECT^*. In the author's paper [9] it is introduced the calculus $\mathsf{MQC} = \mathsf{IQC} + M + ECT^*$, which can be considered as a possible constructive predicate calculus. Note that this calculus is not intermediate between intuitionistic and classical calculi. The arithmetic theory based on the calculus MQC and the Peano axioms is an extension of Markov arithmetic $\mathsf{MA} = \mathsf{HA} + M + ECT$.

3. Absolute realizability. During the study of the predicate logic of recursive realizability, it was revealed that the semantics of predicate formulas under consideration is somewhat occasional. If we add the truth predicate to the language of formal arithmetic, in a natural way extend the concept of recursive realizability to this extended language and define the concept of a realizable predicate formula, then the set of such formulas will significantly narrow. In the paper [10] the author has shown that this procedure can be done up to any constructive ordinal, thus we obtain a transfinite hierarchy of constructive logics. The dependence of predicate logic on the language in which the values of predicate variables are formulated leads to the need to develop constructive semantics that does not depend on this language. It is possible to do this, however, while somewhat avoiding the idea of strict constructiveness and working within the framework of traditional set-theoretic mathematics. In the paper [11] the author introduced the concept of a generalized predicate and the concept of absolute realizability for predicate formulas. At the same time, it turned out that the concepts of absolute irrefutability and uniform absolute realizability are identified. It has been proved that the predicate logic of absolute realizability is Π^1_1-complete. It also turned out that it is quite possible to do without a broad set-theoretic rampage: if the predicate formula is not uniformly absolutely realizable, then there is a refutation of it in the language obtained by adding to the arithmetic language any Π^1_1-complete predicate. On the basis of the concept of a generalized predicate, in the paper [12] the author has developed the elements of constructive model theory, in which constructive logic is combined with the classical theory of constructive models.

4. Predicate logics of constructive theories. Once G. E. Mints asked the author what about the predicate logic based on the translation from the language of arithmetic into the language of arithmetic in all finite types proposed by K. Gödel [13]. Later it turned out that he meant a set-theoretic interpretation of such a translation, and the answer is trivial: such logic coincides with the classical one. However, the author was interested in constructive semantics. Since Gödel's translation is much more complicated than recursive realizability, intuitive approaches do not work here at all. The author had to formalize in a fairly general way the methods of investigating predicate logics based on the application of the Tennenbaum theorem. A constructive arithmetic theory is any extension of the theory $\mathsf{HA} + M + CT$, where CT is the scheme

$$\forall x\, \exists y\, A(x, y) \to \exists z\, \forall x\, \exists y\, (\{z\}(x) = y\, \&\, A(x, y)).$$

If T is an arithmetic theory, then we will call a closed predicate formula T-valid if every closed arithmetic instance of it belongs to the theory T. Call the set of all T-valid formulas the predicate logic of the theory T and denote $\mathcal{L}(T)$. In the paper [14] the author has proved that if T is a constructive arithmetic theory, then $T \leq_1 \mathcal{L}(T)$. For a number of theories, the nonarithmeticity of the corresponding predicate logic is thus established. In fact, the nonarithmeticity result can be extended to a broader class of so-called IS-theories. An extension of the theory HA is called IS-theory if there are such Σ_1-formulas $A(x)$ and $B(x)$ that the formula $\forall x\, \neg(A(x)\, \&\, B(x))\, \&\, \neg\forall x\, \exists y\, ((A(x) \to y = 0)\, \&\, (B(x) \to y \neq 0))$ belongs to this theory In particular, all constructive arithmetic theories are IS-theories. It is proved that if T is IS-theory, then $T^- \leq_1 \mathcal{L}(T)$, where T^- is the negative fragment of the theory of T. The mentioned theorems imply the nonarithmeticity of a number of predicate logics based on the modified realizability introduced by Kreisel [15], and the undecidabilty of some predicate logics based on the Gödel interpretation. The above theorems give lower bounds of the logical complexity of constructive predicate logics. In 1985, V. A. Vardanyan [16] obtained an upper bound on the predicate logic of provability. The ideas he used were successfully applied to the study of predicate logics of the so-called internally enumerable arithmetic theories. In the paper [17] the author has proved that the predicate logic of every internally enumerable IS-theory is Π^T_1-complete. Hence, for example, it follows that the predicate logic of recursive realizability is Π^V_1-complete, where V is the set of all true arithmetic sentences, and also that the predicate logic of Markov arithmetic is Π^2_0-complete.

5. Primitive recursive realizability. It is of interest to consider variants of intuitionistic semantics, in which not the entire class of partial recursive functions is used for the interpretation of effective operations, as in Kleene's recursive realizability, but some of its subclasses. In 1994 Z. Damnanovich [18] introduced the concept of strictly primitive recursive realizability for arithmetic formulas, which combines the ideas of recursive realizability and Kripke models. In 2003 B. H. Park proved in his dissertation that the predicate logic of strictly primitive recursive realizability is nonarithmetic. The proof was essentially based on the claim from [18] that the calculus IQC is sound with respect to strictly primitive recursive realizability. Later, in the paper [19], the author proved the fallacy of that claim. However, the result of Park remains true. The correct proof of it was obtained by the author [20]. Another variant of primitive recursive realizability is proposed by S. Salehi [21]. In the paper [19] the author proves that this concept differs significantly from the concept of strictly primitive recursive realizability. The nonarithmeticity of predicate logic of primitive recursive realizability by Salehi is proved in D. A. Viter's dissertation. His technically complex proof is based on the author's results mentioned above about predicate logics of constructive theories and results of M. Ardeshir [22] on a translation of intuitionistic predicate logic into basic predicate logic. In the paper [23] the author proposed another, ideologically and technically simpler proof of the same result.

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