OPERATIONS ON NON-DETERMINISTIC MATRICES AND THEIR USE

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It is well known that every propositional logic which satisfies certain very natural conditions can be characterized semantically using a multi-valued matrix. However, there are many important decidable logics whose characteristic matrices necessarily consist of an infinite number of truth values. In such a case it might be quite difficult to find any of these matrices, or to use one when it is found. Even in case a logic does have a finite characteristic matrix it might be difficult to discover this fact, or to find such a matrix. The deep reason for these difficulties is that in an ordinary multi-valued semantics the rules and axioms of a system should be considered as a whole, and there is no method for separately determining the semantic effects of each rule alone. In contrast, by allowing the use of non-deterministic operations, one can provide in a lot of cases simple *modular* semantics of axioms and rules of inference, so that the semantics of a system is obtained by joining the semantics of its rules in the most straightforward way. The main tool for this task is the use of finite *non-deterministic matrices (Nmatrices).* Nmatrices differ from (ordinary) matrices in that the truth value of a compound formula may not be uniquely determined by the truth values of its immediate subformulas, but only *constrained* by those truth values. This means that truth values of compound formulas are chosen non-deterministically from a set of options. The particular instance of ordinary matrices is obtained when all these sets are singletons. For some logics, this generalization provides an effective finite-valued semantics, where finite-valued matrices are beyond reach. The use of finite structures of this sort has the benefit of preserving all the advantages of logics with ordinary finite-valued semantics, like decidability and compactness, while it is applicable to a much larger family of logics. Accordingly, since its introduction in [6], the framework of Nmatrices has proven to be very useful, and it has been widely investigated and utilized in various areas, like many-valued logics, paraconsistent logics, and proof theory. (See [5] for a survey of Nmatrices and Chapters 6–8 and 10 of [3] for some applications.)

As hinted above, one very important advantage of using Nmatrices for providing semantics for a logic **L** is that it frequently allows to provide separate semantics to each rule and axiom of **L**, and then get semantics for **L** itself using an appropriate combination of the semantics of its rules and axioms. The basic idea here is that the main effect of a “normal” rule or axiom is to reduce the degree of non-determinism of operations by forbidding some options (in non-deterministic computations of truth values) which we could have had otherwise. Another significant advantage of the semantic framework of Nmatrices is its rich general theory, which includes special useful operations, not available for matrices (or for other types of non-deterministic semantics). Two such operations are expansion and refinement ([2, 1]). Both of these operations transform a given Nmatrix (that may be an ordinary matrix) to another one. The former amounts to a simple duplication of the truth values that are employed in the given Nmatrix, while the latter reduces the amount of non-determinism by taking out possible values from the interpretations of the connectives. The two operations were shown useful for the modular construction of families of paraconsistent logics [1, 4], as well as for studying maximality properties in the constructed logics [2].

In this talk we mainly concentate on an operation we call *rexpansion* (refined expansion) which is a quite useful combination of expansion and refinement. This combined operation proves to be a powerful tool for generating new Nmatrices from existing ones. Properties of this combined operation are presented, along with its effects on the consequence relations which are induced by the operated Nmatrices. In particular, we identify a useful sufficient criterion for a rexpansion of an Nmatrix to result in an equivalent Nmatrix, that induces the same logic.

The main application of rexpansion we present is for the problem of conservatively extending a given logic **L** with new connectives which have some desirable properties. The method is to apply appropriate rexpansion to a matrix (or an Nmatrix) that is known to be characteristic for **L**, getting by this alternative semantics for it, for which the addition of the desired connectives is an easier task. The relations between the original logic and the extended one follow then from the general properties of rexpansions. We demonstrate this method with several examples, including matrices (and Nmatrices) for classical logic, paraconsistent logics, finite-valued logics and infinite-valued logics. The most important demonstration of this technique provides a new (and as we show, significantly better) solution for the problem of constructing *paraconsistent fuzzy logics*. These are logics that are useful for modeling vague propositions, while avoiding the explosion principle, according to which any proposition follows from a contradiction. A first solution to this problem was given in [7], using a completely different approach. However, we show that this solution has some serious drawbacks, which are overcome in the solution proposed here. Our solution is obtained by performing different rexpansions on the Gödel matrix, and then augmenting the resulted Nmatrices with an involutive negation. We further investigate the connection between the various constructed logics.

Finally, in addition to applications of rexpansions of the abovementioned sort, we also show how rexpansions are (implicitely) used in the construction of sequent calculi for many non-classical logics.

References

[1] A. Avron. Non-deterministic semantics for logics with a consistency operator. *Journal of Approximate Reasoning*, 45, 271–287, 2007.

[2] A. Avron, A. Arieli and A. Zamansky. Maximal and premaximal paraconsistency in the framework of three-valued semantics. *Studia Logica*, 99, 31–60, 2011.

[3] A. Avron, A. Arieli and A. Zamansky. **Theory of Effective Propositional Paraconsistent Logics**. Studies in Logic (Mathematical Logic and Foundations) 75, College Publications, 2018.

[4] A. Avron, B. Konikowska and A. Zamansky. Modular Construction of Cut-free Sequent Calculi for Paraconsistent Logics. *Proceedings of Logic in Computer Science (LICS) 27*, 85–94, 2012.

[5] A. Avron and A. Zamansky. Non-deterministic semantics for logical systems – A survey. In **Handbook of Philosophical Logic** (D. Gabbay and F. Guenther, editors), vol. 16, 227–304, Springer, 2011.

[6] A. Avron and I. Lev. Non-deterministic multi-valued structures. *Journal of Logic and Computation*, 15, 241–261, 2005.

[7] R. Ertola, F. Esteva, T. Flaminio, L. Godo and C. Noguera. Paraconsistency properties in degree-preserving fuzzy logics. *Soft Computing*, 19, 531–546, 2015.

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