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Abstracts

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Keynote Talks

OPERATIONS ON NON-DETERMINISTIC MATRICES AND THEIR USE

ARNON AVRON

It is well known that every propositional logic which satisfies certain very natural conditions can be characterized semantically using a multi-valued matrix. However, there are many important decidable logics whose characteristic matrices necessarily consist of an infinite number of truth values. In such a case it might be quite difficult to find any of these matrices, or to use one when it is found. Even in case a logic does have a finite characteristic matrix it might be difficult to discover this fact, or to find such a matrix. The deep reason for these difficulties is that in an ordinary multi-valued semantics the rules and axioms of a system should be considered as a whole, and there is no method for separately determining the semantic effects of each rule alone. In contrast, by allowing the use of non-deterministic operations, one can provide in a lot of cases simple *modular* semantics of axioms and rules of inference, so that the semantics of a system is obtained by joining the semantics of its rules in the most straightforward way. The main tool for this task is the use of finite *non-deterministic matrices (Nmatrices).* Nmatrices differ from (ordinary) matrices in that the truth value of a compound formula may not be uniquely determined by the truth values of its immediate subformulas, but only *constrained* by those truth values. This means that truth values of compound formulas are chosen non-deterministically from a set of options. The particular instance of ordinary matrices is obtained when all these sets are singletons. For some logics, this generalization provides an effective finite-valued semantics, where finite-valued matrices are beyond reach. The use of finite structures of this sort has the benefit of preserving all the advantages of logics with ordinary finite-valued semantics, like decidability and compactness, while it is applicable to a much larger family of logics. Accordingly, since its introduction in [6], the framework of Nmatrices has proven to be very useful, and it has been widely investigated and utilized in various areas, like many-valued logics, paraconsistent logics, and proof theory. (See [5] for a survey of Nmatrices and Chapters 6–8 and 10 of [3] for some applications.)

As hinted above, one very important advantage of using Nmatrices for providing semantics for a logic **L** is that it frequently allows to provide separate semantics to each rule and axiom of **L**, and then get semantics for **L** itself using an appropriate combination of the semantics of its rules and axioms. The basic idea here is that the main effect of a "normal" rule or axiom is to reduce the degree of non-determinism of operations by forbidding some options (in non-deterministic computations of truth values) which we could have had otherwise. Another significant advantage of the semantic framework of Nmatrices is its rich general theory, which includes special useful operations, not available for matrices (or for other types of non-deterministic semantics). Two such operations are expansion and refinement ([2, 1]). Both of these operations transform a given Nmatrix (that may be an ordinary matrix) to another one. The former amounts to a simple duplication of the truth values that are employed in the given Nmatrix, while the latter reduces the amount of non-determinism by taking out possible values from the interpretations of the connectives. The two operations were shown useful for the modular construction of families of paraconsistent logics [1, 4], as well as for studying maximality properties in the constructed logics [2].

In this talk we mainly concentate on an operation we call *rexpansion* (refined expansion) which is a quite useful combination of expansion and refinement. This combined operation proves to be a powerful tool for generating new Nmatrices from existing ones. Properties of this combined operation are presented, along with its effects on the consequence relations which are induced by the operated Nmatrices. In particular, we identify a useful sufficient criterion for a rexpansion of an Nmatrix to result in an equivalent Nmatrix, that induces the same logic.

The main application of rexpansion we present is for the problem of conservatively extending a given logic **L** with new connectives which have some desirable properties. The method is to apply appropriate rexpansion to a matrix (or an Nmatrix) that is known to be characteristic for **L**, getting by this alternative semantics for it, for which the addition of the desired connectives is an easier task. The relations between the original logic and the extended one follow then from the general properties of rexpansions. We demonstrate this method with several examples, including matrices (and Nmatrices) for classical logic, paraconsistent logics, finite-valued logics and infinite-valued logics. The most important demonstration of this technique provides a new (and as we show, significantly better) solution for the problem of constructing *paraconsistent fuzzy logics*. These are logics that are useful for modeling vague propositions, while avoiding the explosion principle, according to which any proposition follows from a contradiction. A first solution to this problem was given in [7], using a completely different approach. However, we show that this solution has some serious drawbacks, which are overcome in the solution proposed here. Our solution is obtained by performing different rexpansions on the Gödel matrix, and then augmenting the resulted Nmatrices with an involutive negation. We further investigate the connection between the various constructed logics.

Finally, in addition to applications of rexpansions of the abovementioned sort, we also show how rexpansions are (implicitely) used in the construction of sequent calculi for many non-classical logics.

References

[1] A. Avron. Non-deterministic semantics for logics with a consistency operator. *Journal of Approximate Reasoning*, 45, 271–287, 2007.

[2] A. Avron, A. Arieli and A. Zamansky. Maximal and premaximal paraconsistency in the framework of three-valued semantics. *Studia Logica*, 99, 31–60, 2011.

[3] A. Avron, A. Arieli and A. Zamansky. **Theory of Effective Propositional Paraconsistent Logics**. Studies in Logic (Mathematical Logic and Foundations) 75, College Publications, 2018.

[4] A. Avron, B. Konikowska and A. Zamansky. Modular Construction of Cut-free Sequent Calculi for Paraconsistent Logics. *Proceedings of Logic in Computer Science (LICS) 27*, 85–94, 2012.

[5] A. Avron and A. Zamansky. Non-deterministic semantics for logical systems – A survey. In **Handbook of Philosophical Logic** (D. Gabbay and F. Guenther, editors), vol. 16, 227–304, Springer, 2011.

[6] A. Avron and I. Lev. Non-deterministic multi-valued structures. *Journal of Logic and Computation*, 15, 241–261, 2005.

[7] R. Ertola, F. Esteva, T. Flaminio, L. Godo and C. Noguera. Paraconsistency properties in degree-preserving fuzzy logics. *Soft Computing*, 19, 531–546, 2015.

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THE POTENTIALITIES OF NON-STANDARD THEORIES OF PROBABILTY

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The Logic of Evidence and Truth LET_F, an extension of the Belnap-Dunn's logic of First-Degree Entailment (FDE), was introduced in [4] and extensively treated in [5]. The intention behind LET_F was to develop an intuitive logical reading for the connections between evidence and truth in terms of preservation of evidence, in a language for a paraconsistent and paracomplete logic enriched with the connective $\circ$ for consistency and $\bullet$ for inconsistency, interpreted respectively as *classicality* (or coherence, or consistency) and *non-classicality* of a proposition.

LET_F is Logic of Formal Inconsistency and Undeterminatedness. One of the most basic tenets of such logics is that contradictions are inconsistent, but not necessarily the other way around: the concept of inconsistency is wider that of the contradiction, and similarly, the concept of consistency is wider than mere non-contradictoriness. This leads to the result that not all contradictions are the same, an idea already voiced by some philosophers.

Although other authors such as E. Mares and G. Priest have thought about paraconsistent probabilities, the first steps on a formal paraconsistent theory of probability based on a Logic of Formal Inconsistency was introduced in [2], investigating notions of conditional probability and paraconsistent updating via versions of Bayes' theorem for conditionalization.

The choice for an apparently weak paraconsistent and paracomplete logic is justified since evidence can be missing, incomplete or even contradictory. However, LET_F is only apparently weak, as it fully restores all classical reasoning in the presence of the operator for classicality.

A probabilistic semantics developed on top of LET_F permits to measure and quantify the degree of evidence attributed to a proposition. In this way a probability measure P on LET_F quantifies the amount $P(\alpha)$ of evidence attributed to a proposition α. Not only this, but the connective $\circ$ of classicality that is part of the language of LET_F permits to qualify the degree of confidence on the evidence for a proposition α.

When $\circ\alpha$ holds excluded middle and explosion are valid, that is: $\alpha, \neg\alpha, \circ\alpha \vdash \beta$ although $\alpha, \neg\alpha \nvdash \beta$, and $\circ\alpha \vdash \alpha \vee \neg\alpha$, while $\nvdash \alpha \vee \neg\alpha$. The connective $\bullet\alpha$, defined as $\neg \circ \alpha$, acts as a non-classicality operator.

LET_F is characterized (in terms of soundness and completeness) by a valuation semantics which also provides a decision procedure for LET_F(cf. [5]). Kripke-style models for LET_F and for the logic LET_J(which extends Nelson's logic $N4$) appear in [1]. The models represent a database that receives information as time passes, and such information A can be positive, negative, non-reliable, or reliable, while a formula $\circ A$ means that the information about A, either positive or negative, is reliable. This proposal is in line with the interpretation of FDE and $N4$ as information-based logics,

The option for LET_F is justified since LET_F is a paraconsistent and paracomplete logic, and thus agents under this logic can believe in contradictions and at the same time are not obliged to believe in all classical tautologies, maintaining rationality even in incomplete or contradictory scenarios.

A second aspect connected to LET_F is the interesting possibility of enlarging K. Popper's notion of autonomous probability, in order to obtain a new version of paracomplete and paraconsistent autonomous probability theory where Kolmogorovian probabilities can be obtained as a particular case. The main intention is to obtain a probability theory which is able to deal with contradictory events , at the same time avoiding the philosophical criticisms about the Kolmogorovian conditional probability.

A well-known problem involving the familiar conditional probability as a ratio formula is that it represents a barrier for many applications of probability in view of the so-called problem of zero-probability. Even if it is a consequence of the definition of standard probability theory that propositions representing contradictory events have zero probability (or in other words, classically impossible events have zero probability), the converse is not true– events with probability zero are not impossible. There are several examples, illustrating this point, which affects directly the classical definition of conditional probability as a ratio formula, since it excludes conditional probabilities with zero antecedents: $P(A|B) = def\,\frac{P(A \wedge B)}{P(B)}$, provided $P(B) \neq 0$.

I intend to discuss the ideas and the prospects of a LET_F- based probability theory, as well as the development of a new form of autonomous Popperian probability theory that circumvents the problem of zero-probability, following the direction of Popper's philosophy and taking into account that neither Kolmogorov's nor Popper's approach deal with missing evidence (information gaps) nor with logically conflicting situations (information gluts).

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References

[1] Kripke-style models for Logics of Evidence and Truth. H. Antunes nd W. A. Carnielli and A. Kapsner and A. Rodrigues *Axioms*, 9 (3), 2020 Open access at: https://epub.ub.uni-muenchen.de/75403/1/axioms-09-00100-v3.pdf

[2] J. Bueno-Soler and W. A. Carnielli. Paraconsistent probabilities: consistency, contradictions and Bayes' Theorem. *Entropy*, 18(9), 2016. Open access at: http://www.mdpi.com/1099-4300/18/9/325/htm

[3] W. A. Carnielli and M. E. Coniglio. Paraconsistent Logic: Consistency, Contradiction and Negation. Springer, 2016.

[4] W. A. Carnielli and A. Rodrigues. An epistemic approach to paraconsistency: a logic of evidence and truth. *Synthese*, 196, 3789-3813, 2017.

[5] A. Rodrigues and J. Bueno-Soler and W.A. Carnielli. Measuring evidence: a probabilistic approach to an extension of Belnap–Dunn logic. *Synthese*, 198, 5451–5480, 2021.

[6] W. A. Carnielli and J. Bueno-Soler. Where the truth lies: a paraconsistent approach to Bayesian epistemology. *Logica Universalis*, to appear.

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LOGIC OF COMBINATORY LOGIC

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Models of computation can be broadly classified into three categories: sequential models, functional models and concurrent models. *Combinatory logic* (CL) is a well-known functional model of computation which accomplishes the idea to eliminate the need for quantified variables in mathematical logic. It is based on combinators, which are higher order functions, and uses only function application. Combinators where invented a century ago, in the early 1920s, by Moses Schönfinkel. The foundations of combinatory logic were established by Haskell Curry in the 1930s and it has been developing ever since. A recent comprehensive overview can be found in [6]. CL is of the same expressive power as the λ-calculus, the functional model of computation introduced by Alonso Church in the 1930s. With respect to this CL captures all computable functions without using bound variables and the obstacles that emerge from bound variables in λ-calculus are completely avoided.

Typed Combinatory logic is a system in which application is controlled by types assigned to terms. Following the first type system of CL, called *simply types*, various type systems emerged, such as intersection types, dependent types, polymorphic types found their primary applications in programming languages, automated theorem provers, proof assistants, program synthesis. Over time the range of applications has been widened to machine learning, artificial intelligence, cognitive representation, natural language and physics. New applications urge for further research and development of the theory and reasoning about CL both typed and untyped.

Various extensions of combinatory logic, both untyped and typed, have been considered in order to obtain formalisms capable to express new features and paradigms. The extension of the syntax with new constructors such as pairs, records, variants, among others, enables building compound data structures, better organization of data and dealing with heterogeneous collections of values. Extending the syntax with a new operator such as a probabilistic operator shifts the computation to a new paradigm, called probabilistic computation. Herein, the approach we are interested in combines typed combinatory logic with classical propositional logic.

In this talk we discuss and present an extension of the simply typed combinatory logic which is a classical propositional logic for reasoning about combinatory logic. It is called *Logic of Combinatory logic*, denoted by LCL and introduced in [4]. We define its syntax, axiomatic system and semantics. The logic LCL, can be explained in two ways:

- LCL is a logic obtained by extending the simply typed combinatory logic with classical propositional connectives, and corresponding axioms and rules;
- LCL is a logic obtained from classical propositional logic by replacing propositional letters with type assignment statements $M : \sigma$, i.e. typed combinatory terms, where M is a term of the combinatory logic, and σ is a simple type.

The LCL *axiomatic system* has two kinds of axioms:

- non-logical axioms concerned with features of combinatory logic;
- logical axioms concerned with logical connectives.

The LCL *semantics* is based on the notion of applicative structures extended with special elements corresponding to the primitive combinators. It is inspired by the applicative structures introduced in [5], [3] and [2].

As a first result we prove the *soundness and completeness of the equational theory of untyped CL* with respect to the proposed semantics. As a second result we show the *soundness and completeness of the LCL axiomatization* with respect to the proposed semantics. The completeness theorem is proved by an adaptation of the Henkin-style completeness method. We first prove that every consistent set can be extended to a maximal consistent set, which is further used to introduce the notion of a canonical model. We prove then that every consistent set is satisfiable. As a consequence we have the completeness theorem: whenever a formula is a semantical consequence of a set of formulas, it is also a deductive consequence of that set. Further, we prove soundness and completeness of CL, which yields a *new semantics of* CL. In addition, we prove that LCL is a conservative extension of the simply typed CL.

We will conclude the talk with discussing related work ([1]) and possible applications in automated reasoning and knowledge representation.

The present talk is based on joint work with Simona Kašterović ([4]).

References

[1] Michael Beeson. Lambda logic. IJCAR 2004, volume 3097 of *Lecture Notes in Computer Science*, pages 460–474, 2004.

[2] Silvia Ghilezan, Simona Kašterović. Semantics for combinatory logic with intersection types. *Frontiers in Computer Science*, volume 4, 2022. DOI: https://doi.org/10.3389/fcomp.2022.792570

[3] Simona Kašterović, Silvia Ghilezan. Kripke semantics and completeness for full simply typed lambda calculus. *Journal of Logic and Computation Volume* 30, issue 8: 1567–1608, 2020. DOI: https://doi.org/10.1093/logcom/exaa055

[4] Simona Kašterović, Silvia Ghilezan. Logic of Combinatory Logic. *C*oRR, abs/2212.06675, 2022. DOI: https://doi.org/10.48550/arXiv.2212.06675

[5] John C. Mitchell and Eugenio Moggi. Kripke-style models for typed lambda calculus. *Annals of Pure and Applied Logic*, 51(1-2):99–124, 1991.

[6] Stephen Wolfram. *Combinators: A Centennial View*. Wolfram Media, Incorporated, 2021.

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CONSTRUCTIVE INTERPRETATIONS OF LOGICAL AND LOGICAL-MATHEMATICAL LANGUAGES

VALERII PLISKO

A survey of the author's main results obtained during his lifetime is offered in the context of research in the field of constructive logic conducted in the USSR and Russia (and not only) in the last 50 years.

1. Propositional logic of recursive realizability. By constructive semantics we will understand intuitionistic interpretations of logical and logical-mathematical languages, in which the concept of an effective operation is explained through the concept of an algorithm. The first semantics of this kind called recursive realizability was proposed by S. C. Kleene for the formal arithmetic language. On its basis, various variants of the concept of a realizable predicate (in particular, propositional) formula are possible, of which irrefutability and effective realizability deserve the most attention. A predicate formula is called irrefutable if the universal closure of any arithmetic example of this formula is realizable. A closed predicate formula is called effectively realizable if there exists an algorithm that allows for any closed arithmetic instance of this formula to find its realization. The author proved [1] that in general these concepts do not coincide, but for propositional formulas the question of their coincidence or difference remains open.

In 1932 A. N. Kolmogorov proposed an informal interpretation of intuitionistic logic as the logic of problems. 30 years later, this idea was somewhat refined by Medvedev in the form of the concept of finite validity for propositional formulas. In 1963, Medvedev [2] published an erroneous proof of the theorem that every realizable propositional formula is finitely valid. The author has constructed a counterexample to this statement. In this case, the finite model property of the Medvedev logic and Yankov's characteristic formulas [3] were significantly used. A detailed exposition of this result is available in [4]. Medvedev proved the completeness of fragments of the intuitionistic propositional calculus IPC without negation and disjunction with respect to finite validity. From the above-mentioned erroneous theorem, a similar result was obtained for the propositional logic of recursive realizability. The author proved that the completeness of these fragments of the calculus IPC with respect to recursive realizability really holds. A detailed proof is available in [5].

2. Predicate logic of recursive realizability. The propositional logic of recursive realizability is difficult to investigate. The situation is much better with the predicate logic of recursive realizability. The author proved [6] that the set of all realizable predicate formulas is not arithmetic. The proof was based on the Tennenbaum theorem that there are no recursive non-standard models of arithmetic. The technique developed at the same time allowed solving many issues related to the predicate logic of recursive realizability. The scheme theorem is technically important. The concept of a scheme over the arithmetic language was introduced by M. M. Kipnis [7]. A scheme is a formula in the mixed language of arithmetic and predicate logic. For schemes, all concepts of realizability are introduced by analogy with predicate formulas. The scheme theorem [1] states that for any scheme it is possible to efficiently construct a predicate formula that is realizable in one sense or another if and only if the original scheme is realizable in the same sense. This made it possible to show the difference between the mentioned variants of realizability for predicate formulas, as well as their difference from the concept of uniform realizability, meaning the existence of a single realization for all closed arithmetic instances of a closed predicate formula.

It follows from the Nelson theorem that the intuitionistic predicate calculus IQC is sound with respect to recursive realizability. It was shown by Rose that the calculus IPC is not complete with respect to this semantics. Subsequently, Markov formulated a logical principle, now called the Markov principle, meaning, in particular, the realizability of the predicate formula

$$\forall x\,(P(x) \vee \neg P(x)) \to (\neg\neg\exists x\, P(x) \to \exists x\, P(x))$$

(denote it M), non-deducible in IQC. A scheme ECT over the arithmetic language, called the extended Church thesis, is sound with respect to the semantics of recursive realizability. In the presence of Markov's principle, this scheme is equivalent to the scheme

$$\forall x\,(\neg A(x) \to \exists y\, B(x,y)) \to \exists z\, \forall x\,(\neg A(x) \to \exists y\,(\{z\}(x) = y \,\&\, B(x,y))),$$

where $\{z\}$ denotes a partial recursive function with a Gödel number z. The scheme theorem allows us to replace this scheme with a predicate formula; we denote it ECT^*. In the author's paper [9] it is introduced the calculus $\mathsf{MQC} = \mathsf{IQC} + M + ECT^*$, which can be considered as a possible constructive predicate calculus. Note that this calculus is not intermediate between intuitionistic and classical calculi. The arithmetic theory based on the calculus MQC and the Peano axioms is an extension of Markov arithmetic $\mathsf{MA} = \mathsf{HA} + M + ECT$.

3. Absolute realizability. During the study of the predicate logic of recursive realizability, it was revealed that the semantics of predicate formulas under consideration is somewhat occasional. If we add the truth predicate to the language of formal arithmetic, in a natural way extend the concept of recursive realizability to this extended language and define the concept of a realizable predicate formula, then the set of such formulas will significantly narrow. In the paper [10] the author has shown that this procedure can be done up to any constructive ordinal, thus we obtain a transfinite hierarchy of constructive logics. The dependence of predicate logic on the language in which the values of predicate variables are formulated leads to the need to develop constructive semantics that does not depend on this language. It is possible to do this, however, while somewhat avoiding the idea of strict constructiveness and working within the framework of traditional set-theoretic mathematics. In the paper [11] the author introduced the concept of a generalized predicate and the concept of absolute realizability for predicate formulas. At the same time, it turned out that the concepts of absolute irrefutability and uniform absolute realizability are identified. It has been proved that the predicate logic of absolute realizability is Π^1_1-complete. It also turned out that it is quite possible to do without a broad set-theoretic rampage: if the predicate formula is not uniformly absolutely realizable, then there is a refutation of it in the language obtained by adding to the arithmetic language any Π^1_1-complete predicate. On the basis of the concept of a generalized predicate, in the paper [12] the author has developed the elements of constructive model theory, in which constructive logic is combined with the classical theory of constructive models.

4. Predicate logics of constructive theories. Once G. E. Mints asked the author what about the predicate logic based on the translation from the language of arithmetic into the language of arithmetic in all finite types proposed by K. Gödel [13]. Later it turned out that he meant a set-theoretic interpretation of such a translation, and the answer is trivial: such logic coincides with the classical one. However, the author was interested in constructive semantics. Since Gödel's translation is much more complicated than recursive realizability, intuitive approaches do not work here at all. The author had to formalize in a fairly general way the methods of investigating predicate logics based on the application of the Tennenbaum theorem. A constructive arithmetic theory is any extension of the theory $\mathsf{HA} + M + CT$, where CT is the scheme

$$\forall x\, \exists y\, A(x, y) \to \exists z\, \forall x\, \exists y\, (\{z\}(x) = y \,\&\, A(x, y)).$$

If T is an arithmetic theory, then we will call a closed predicate formula T-valid if every closed arithmetic instance of it belongs to the theory T. Call the set of all T-valid formulas the predicate logic of the theory T and denote $\mathcal{L}(T)$. In the paper [14] the author has proved that if T is a constructive arithmetic theory, then $T \leq_1 \mathcal{L}(T)$. For a number of theories, the nonarithmeticity of the corresponding predicate logic is thus established. In fact, the nonarithmeticity result can be extended to a broader class of so-called IS-theories. An extension of the theory HA is called IS-theory if there are such Σ_1-formulas $A(x)$ and $B(x)$ that the formula $\forall x\, \neg(A(x) \,\&\, B(x)) \,\&\, \neg\forall x\, \exists y\, ((A(x) \to y = 0) \,\&\, (B(x) \to y \neq 0))$ belongs to this theory In particular, all constructive arithmetic theories are IS-theories. It is proved that if T is IS-theory, then $T^- \leq_1 \mathcal{L}(T)$, where T^- is the negative fragment of the theory of T. The mentioned theorems imply the nonarithmeticity of a number of predicate logics based on the modified realizability introduced by Kreisel [15], and the undecidabilty of some predicate logics based on the Gödel interpretation. The above theorems give lower bounds of the logical complexity of constructive predicate logics. In 1985, V. A. Vardanyan [16] obtained an upper bound on the predicate logic of provability. The ideas he used were successfully applied to the study of predicate logics of the so-called internally enumerable arithmetic theories. In the paper [17] the author has proved that the predicate logic of every internally enumerable IS-theory is Π^T_1-complete. Hence, for example, it follows that the predicate logic of recursive realizability is Π^V_1-complete, where V is the set of all true arithmetic sentences, and also that the predicate logic of Markov arithmetic is Π^2_0-complete.

5. Primitive recursive realizability. It is of interest to consider variants of intuitionistic semantics, in which not the entire class of partial recursive functions is used for the interpretation of effective operations, as in Kleene's recursive realizability, but some of its subclasses. In 1994 Z. Damnanovich [18] introduced the concept of strictly primitive recursive realizability for arithmetic formulas, which combines the ideas of recursive realizability and Kripke models. In 2003 B. H. Park proved in his dissertation that the predicate logic of strictly primitive recursive realizability is nonarithmetic. The proof was essentially based on the claim from [18] that the calculus IQC is sound with respect to strictly primitive recursive realizability. Later, in the paper [19], the author proved the fallacy of that claim. However, the result of Park remains true. The correct proof of it was obtained by the author [20]. Another variant of primitive recursive realizability is proposed by S. Salehi [21]. In the paper [19] the author proves that this concept differs significantly from the concept of strictly primitive recursive realizability. The nonarithmeticity of predicate logic of primitive recursive realizability by Salehi is proved in D. A. Viter's dissertation. His technically complex proof is based on the author's results mentioned above about predicate logics of constructive theories and results of M. Ardeshir [22] on a translation of intuitionistic predicate logic into basic predicate logic. In the paper [23] the author proposed another, ideologically and technically simpler proof of the same result.

References

[1] V. E. Plisko. Some variants of the notion of realizability for predicate formulas (Russian). *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 42(3), 637–653, 1978. Translation *Math. USSR Izvestiya*, 12, 588–604.

[2] Yu. T. Medvedev. Interpretation of logical formulae by means of finite problems and its relation to the realizability theory (Russian). *Doklady Akademii Nauk SSSR*, 148(4), 771–774, 1963. Translation *Soviet Math. Dokl.*, 4, 180–183.

[3] V. A. Jankov. On the connection between deducibility in the intuitionistic propositional calculus and finite implicative structures (Russian). *Doklady Akademii Nauk SSSR*, 151(6), 1293–1294, 1963. Translation *Soviet Math. Dokl.*, 4, 1203–1204.

[4] V. Plisko. An Application of the Yankov characteristic formulas. *V.A. Yankov on Non-Classical Logics, History and Philosophy of Mathematics. Outstanding Contributions to Logic*, 24, 209–219, 2022.

[5] V. Plisko. A survey of propositional realizability logic. *Bulletin of Symbolic Logic*, 15(1), 1–42, 2009.

[6] V. E. Plisko. The nonarithmeticity of the class of realizable predicate formulas (Russian). *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 41(3), 483–502, 1977. Translation *Math. USSR Izvestiya*, 11, 453–471.

[7] M. M. Kipnis. Invariant properties of systems of formulas of elementary axiomatic theories (Russian). *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 34(5), 963–976, 1970. Translation *Math. USSR Izvestiya*, 4, 965–978.

[8] G. F. Rose. Propositional calculus and realizability. *Transactions of the American Mathematical Society*, 75, 1–19, 1953.

[9] V. E. Plisko. On certain constructive predicate calculus. arXiv:2009. 10940, 1–30, 2022.

[10] V. Plisko. Transfinite sequences of constructive predicate logics. *Computer Science — Theory and Applications. Lecture Notes in Computer Science*, 6072, 315–326, 2010.

[11] V. E. Plisko. Absolute realizability of predicate formulas (Russian). *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 47(2), 315–334, 1983. Translation *Math. USSR Izvestiya*, 22, 291–308.

[12] V. E. Plisko. The elements of the constructive model theory (Russian). *Fundamentalnaya i Prikladnaya Matematika*, 8(3), 783–828, 2002.

[13] K. Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *Dialectica*, 12(3/4), 280–287, 1958.

[14] V. E. Plisko. Constructive formalization of Tennenbaum's theorem and its applications. *Matematicheskie Zametki*, 48(3), 108–118, 1990. Translation *Mathematical Notes*, 48, 950–957.

[15] G. Kreisel. Interpretation of analysis by means of constructive functionals of finite types. *Constructivity in Mathematics*, 101–128. North-Holland, Amsterdam, 1959.

[16] V. A. Vardanyan. *On predicate provability logic (Russian).* Preprint. Moscow, Scientific Council on the Complex Problem "Cybernetics", 1985.

[17] V. E. Plisko. On arithmetic complexity of certain constructive logics. *Matematicheskie Zametki*, 52(1), 94–104, 1992. Translation *Mathematical Notes*, 52, 701–709.

[18] Z. Damnjanovic. Strictly primitive recursive realizability. I. *Journal of Symbolic Logic*, 59(4), 1210–227, 1994.

[19] V. Plisko On primitive recursive realizabilities. *Computer Science — Theory and Applications. Lecture Notes in Computer Science*, 3967, 304–312, 2006.

[20] V. Plisko The nonarithmeticity of the predicate logic of strictly primitive recursive realizability *The Review of Symbolic Logic*, 15(3), 693–721, 2022.

[21] S. Salehi. Provably total functions of Basic Arithmetic. *Mathematical Logic Quarterly*, 49(3), 316–322, 2003.

[22] M. Ardeshir. A translation of intuitionistic predicate logic into basic predicate logic. *Studia logica*, 62, 341–352, 1999.

[23] V. E. Plisko. The nonarithmeticity of the predicate logic of primitive recursive realizability(Russian). *Izvestiya RAN. Seriya Matematicheskaya*, 87(2), 196–228, 2023.

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COMPUTATIONAL COMPLEXITY OF THEORIES OF RESIDUATED STRUCTURES

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The present talk is based on joint work with C. J. Van Alten [5, 8].

Residuated structures are used as algebraic models of a wide variety of propositional logics; they play particularly prominent role in the study of substructural logics. Equational and atomic theories of such structures correspond to sequents provable in logics; quasi-equational and Horn theories correspond to statements of derivability of sequents from sets of sequents; universal theories correspond to inferences of Boolean combinations of sequents from Boolean combinations of sequents.

In this talk, we are concerned with the following types of residuated structures. A *residuated ordered groupoid* (*rog*, for short) is a tuple $\langle A, \leqslant, \circ, \backslash, /, \rangle$, where $\langle A, \leqslant \rangle$ is a partially ordered set, and $\circ$, $\backslash$, and $/$ are binary operations on A satisfying the residuation condition:

$$a \circ b \leqslant c \iff b \leqslant a \backslash c \iff a \leqslant c/b. \tag{1}$$

The class of all rogs shall be denoted by $\mathcal{ROG}$. A *residuated distributive lattice-ordered groupoid* (*rdg*, for short) is a tuple $\langle A, \vee, \wedge, \circ, \backslash, / \rangle$, where $\langle A, \vee, \wedge \leqslant \rangle$ is a distributive lattice, and $\circ$, $\backslash$, and $/$ are binary operations on A satisfying (1). The class of all rdgs shall be denoted by $\mathcal{RDG}$.

Theories of rogs shall be stated in a first-order language in the signature with operation symbols $\circ$, $\backslash$, and $/$, and the relational symbol $\leqslant$. Theories of rdgs shall be stated in a first-order language in the signature with operation symbols $\vee$, $\wedge$, $\circ$, $\backslash$, $/$, and the relational symbol $=$ (the relational symbol $\leqslant$ is definable in the standard way: $a \leqslant b := a \wedge b = a$). For both rogs and rdgs, terms and valuations are defined in the standard way. The evaluation of formulas and validity are defined as in the standard model theory.

The *atomic theory* of $\mathcal{ROG}$ is the set of the atomic formulas (i.e., expressions of the form $s \leqslant t$) valid in $\mathcal{ROG}$. The *Horn theory* of $\mathcal{ROG}$ is the set of formulas of the form $\alpha_1 \dot{\wedge} \ldots \dot{\wedge} \alpha_n \Rightarrow \alpha$, where $\alpha_1, \ldots, \alpha_n$ and α are all atomic, valid in $\mathcal{ROG}$. The *universal theory* of $\mathcal{ROG}$ is the set of formulas $\forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$, where φ is a Boolean combination of atomic formulas, valid in $\mathcal{ROG}$. The *equational theory* of $\mathcal{RDG}$ is the set of equations valid in $\mathcal{RDG}$. The *quasi-equational theory* of $\mathcal{RDG}$ is the set of quasi-equations (i.e., expressions of the form $\alpha_1 \dot{\wedge} \ldots \dot{\wedge} \alpha_n \Rightarrow \alpha$, where $\alpha_1, \ldots, \alpha_n$ and α are all equations) valid in $\mathcal{RDG}$. The *universal theory* of $\mathcal{RDG}$ is the set of formulas $\forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$, where φ is a Boolean combination of equations, valid in $\mathcal{RDG}$. It was established by Aarts and Trautwein [1] that the atomic theory of $\mathcal{ROG}$, and by Buszkowski [2] that the Horn theory of $\mathcal{ROG}$, are polynomial-time decidable. We present here results on the complexity of universal theory of $\mathcal{ROG}$, as well as universal and quasi-equational theories of $\mathcal{RDG}$:

Theorem 1. *The universal theory of $\mathcal{ROG}$ is coNP-complete.*

Theorem 2. *The universal and quasi-equational theories of $\mathcal{RDG}$ are both EXPTIME-complete.*

The upper bounds are obtained through the use of partial algebras [9, 6, 7]. We show that, if a quantifier-free formula φ is satisfiable in a rog, this is witnessed by a partial rog of size polynomial in the size of φ; moreover, we identify a set of polynomial-time verifiable structural conditions ensuring that a partial structure is a partial rog. This gives us a non-deterministic polynomial-time algorithm for checking satisfiability of quantifier-free formulas in $\mathcal{ROG}$: we guess a partial structure of size polynomial in the size of the input formula φ, check that this structure is a partial rog, and lastly check if φ is satisfied in the structure we guessed. Analogously, we show that, if a quantifier-free formula φ is satisfiable in a rdg, this is witnessed by a partial rdg of size exponential in the size of φ; moreover, there exists a deterministic algorithm for checking if such a partial rdg exists for a formula (the algorithm is based on a structural characterization of partial rdgs using exponential-time verifiable properties of partial structures). In the structural characterization of partial rogs and partial rdgs we essentially rely on the relational frame theory developed by Dunn [3, 4].

The lower bound for Theorem 1 follows from a simple observation that the universal theory of a class of non-trivial structures is as computationally hard the Boolean logic. The lower bound for Theorem 2 is obtained by reduction from a corridor tiling game through satisfiability in a modal logic with the universal modality.

References

[1] E. Aarts and K. Trautwein. Non-associative Lambek categorial grammar in polynomial time. *Mathematical Logic Quarterly*, 41:476–484, 1995.

[2] W. Buszkowski. Lambek Calculus with Nonlogical Axioms. Claudia Casadio and Philip J. Scott and Robert A.G. Seely (eds.) *Language and Grammar: Studies in Mathematical Linguistics and Natural Language*, Center for the Study of Language and Information, 2005, 77–94.

[3] J. M. Dunn. Gaggle theory: An abstraction of Galois connections and residuation, with applications to negation, implication, and various logical operators. In J. van Eijck, editor, *Logics in AI. JELIA 1990*, volume 478 of *Lecture Notes in Computer Science*. Springer, 1990.

[4] J. M. Dunn. Partial gaggles applied to logics with restricted structural rules. Schroeder-Heister P, Došen K (eds) *Substructural logics*, Studies in Logic and Computation, vol 2, Clarendon Press, pp 72–108

[5] D. Shkatov and C. J. Van Alten. Complexity of the universal theory of bounded residuated distributive lattice-ordered groupoids. *Algebra Universalis*, 80(3):36, 2019.

[6] D. Shkatov and C. J. Van Alten. Complexity of the universal theory of modal algebras. *Studia Logica*, 108(2):221–237, 2020.

[7] D. Shkatov and C. J. Van Alten. Computational complexity for bounded distributive lattices with negation. *Annals of Pure and Applied Logic*, 172(7):102962, 2021.

[8] D. Shkatov and C. J. Van Alten. Complexity of the universal theory of residuated ordered groupoids. *Journal of Logic, Language and Information*, 2023. https://doi.org/10.1007/s10849-022-09392-9.

[9] C. J. Van Alten. Partial algebras and complexity of satisfiability and universal theory for distributive lattices, Boolean algebras and Heyting algebras. *Theoretical Computer Science*, 501(82–92), 2013.

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Contributed Talks

ALGORITHMIC COMPLEXITY OF MONADIC MULTIMODAL PREDICATE LOGICS WITH EQUALITY OVER FINITE KRIPKE FRAMES

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1. Introduction

Monadic modal and superintuitionistic logics are, as a rule, undecidable in very poor vocabularies—in most cases, to prove undecidability, it suffices to use a single monadic predicate letter and two or three individual variables [9, 11, 12, 13, 14, 8] (for undecidability of related fragments of the classical logics, see [17, 10, 6, 7]). At the same time, the monadic fragment with equality of the classical predicate logic $\mathbf{QCl}^{=}$ is decidable [1]. Hence, it is of interest to identify settings where decidability can be obtained.

Proofs of undecidability of monadic fragments usually rely on the so-called the "Kripke trick" [4], a simulation of a subformula $P(x,y)$ of a classical formula with a monadic modal formula $\Diamond(Q_1(x)\wedge Q_2(y))$. Hence, indentifying decidable fragments involves discovering setting where the Kripke trick is not applicable. This has been done syntactically by Wolter and Zakharyaschev [18], who discovered monodic fragments (note that these differ from monadic fragments) disallowing the application of modalities to formulas with more than one parameter. Here, we consider a simple semantical setting where the Kripke trick does not work: the monadic predicate logic with equality of a Kripke frame with finitely many possible worlds (but, possibly, infinite domains). We also obtain precise complexity bounds for monadic logics of classes of Kripke frames with finitely many possible worlds. This is of interest since precise bounds beyond Σ^0_1 hardness are scarce in the literature on predicate modal logic. The observations presented here are generalizations to the multimodal settings of results from in [5, 15].

2. Preliminaries

We consider the n-modal, where $n \in \mathbb{N}^+$, predicate language $\mathcal{L}_n$ obtained by adding to the classical predicate language $\mathcal{L}_0$ unary modal connectives $\Box_1, \ldots, \Box_n$, as well as the language $\mathcal{L}_n^{=}$ obtained by adding to $\mathcal{L}_n$ a designated binary predicate letter $=$. The definitions of formulas are standard. A *monadic $\mathcal{L}_n$-formula* contains only monadic predicate letters. A *monadic $\mathcal{L}_n$-formula with equality* contains only monadic predicate letters and $=$.

By a normal n-modal predicate logic we mean a set of $\mathcal{L}_n$-formulas including the classical predicate logic $\mathbf{QCl}$ and the minimal normal n-modal propositional logic $\mathbf{K}_n$ and closed under Modus Ponens, Substitution, Necessitation, and Generalisation. A normal n-modal predicate logic with equality additionally contains the classical equality axioms. The minimal logic containing $\mathbf{QCl}$ and the n-modal propositional logic L is denoted by $\mathbf{Q}L$; the minimal extension of $\mathbf{Q}L$ containing the classical equality axioms is denoted by $\mathbf{Q}^{=}L$. The minimal extension of an n-modal predicate logic L containing, for each $k \in \{1, \ldots, n\}$, the Barcan formula $\boldsymbol{bf}_k = \forall x\, \Box_k P(x) \to \Box_k \forall x\, P(x)$, is denoted by $L.\mathbf{bf}$.

A *fusion* of 1-modal propositional logics $L_1, \ldots, L_n$ is the logic $L_1 * \ldots * L_n = \mathbf{K}_n \oplus (L_1 \cup L_2' \cup \ldots \cup L_n')$, where L_i' is obtained from L_i by replacing every occurrence of $\Box_1$ with $\Box_i$.

We use the framework of Kripke semantics for logics with and without equality (for more details, see [3]; our terminology differs from that adopted in [3]). There are two natural way to extend the well-known Kripke semantics for logics without equality to logics with equality; to treat equality as identity or as hereditary congruence. Unlike the classical logic, these two treatments of equality are not equivalent: the formula $x \neq y \to \Box_k(x \neq y)$ is valid if $=$ is interpreted as identity, but not valid if $=$ is interpreted as hereditary congruence. Here, except in Section 4, we treat equality as congruence.

A Kripke n-frame is a tuple $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$, where W is a non-empty set of worlds and $R_1, \ldots, R_n$ are binary accessibility relations on W. An augmented n-frame is a tuple $\boldsymbol{\mathfrak{F}} = \langle \mathfrak{F}, D \rangle$, where $\mathfrak{F}$ is a Kripke n-frame and D a family $(D_w)_{w\in W}$ of non-empty domains satisfying the expanding domains condition: for every $w, v \in W$,

$$(E) \quad wR_kv \implies D_w \subseteq D_v.$$

The condition (E) is required for soundness and completeness of predicate modal logics whose $\mathcal{L}_0$-fragment is **QCl**. If an augmented n-frame satisfies

$$(C) \quad wR_kv \implies D_w = D_v,$$

then it is called a locally constant augmented n-frame. A model is a tuple $\mathfrak{M} = \langle \boldsymbol{\mathfrak{F}}, I\rangle$, where $\boldsymbol{\mathfrak{F}}$ is an augmented n-frame and I is a family $(I_w)_{w\in W}$ of interpretations of predicate letters: $I_w(P) \subseteq D_w^m$, for every m-ary letter P.

An augmented n-frame with equality is a tuple $\boldsymbol{\mathfrak{F}} = \langle \mathfrak{F}, D, \equiv\rangle$, where $\langle \mathfrak{F}, D\rangle$ is an augmented n-frame and $\equiv$ is a family $(\equiv_w)_{w\in W}$ of equivalence relations, with $\equiv_w \subseteq D_w^2$ whenever $w \in W$, satisfying the heredity condition: for every $w, v \in W$,

$$(H) \quad wR_kv \implies \equiv_w \subseteq \equiv_v.$$

The condition (H) corresponds to the formula $x = y \to \Box_k(x = y)$, which belongs to $\mathbf{Q^{=}K}$, and hence to every normal modal predicate logic with equality. A model with equality is a tuple $\mathfrak{M} = \langle \boldsymbol{\mathfrak{F}}, I\rangle$, where $\boldsymbol{\mathfrak{F}}$ is an augmented n-frame with equality and I is a family $(I_w)_{w\in W}$ of interpretations of predicate letters such that $\equiv_w$ is a congruence on the classical model $M_w = \langle D_w, I_w\rangle$.

The truth relation for $\mathcal{L}_n$ and $\mathcal{L}_n^{=}$ is defined by usual way; in particular, if $a, b \in D_w$ and $\bar{c}$ is a list of elements of D_w of a suitable length, then

$$\begin{array}{lcl}
\mathfrak{M}, w \models a = b & \leftrightharpoons & a \equiv_w b;\\
\mathfrak{M}, w \models P(\bar{c}) & \leftrightharpoons & \bar{c} \in I_w(P);\\
\mathfrak{M}, w \models \forall x\, \varphi(x, \bar{c}) & \leftrightharpoons & \mathfrak{M}, w \models \varphi(d, \bar{c}), \text{ for every } d \in D_w;\\
\mathfrak{M}, w \models \Box_k\varphi(\bar{c}) & \leftrightharpoons & \mathfrak{M}, v \models \varphi(\bar{c}), \text{ for every } v \in R_k(w).
\end{array}$$

The following definitions concern both $\mathcal{L}_n$ and $\mathcal{L}_n^{=}$; for the latter, all the models and augmented frames should be understood as those with equality. A formula φ is true at a world w if a universal closure of φ is true at w. A formula φ is true in a model $\mathfrak{M}$ if φ true at every world of $\mathfrak{M}$; φ is valid on an augmented n-frame $\boldsymbol{\mathfrak{F}}$ if it is true in every model over $\boldsymbol{\mathfrak{F}}$; φ is valid on a Kripke n-frame $\mathfrak{F}$ if φ is valid on every augmented n-frame over $\mathfrak{F}$; φ is valid on a class $\mathscr{C}$ of augmented frames if it is valid on every augmented frame from $\mathscr{C}$.

If $\mathscr{C}$ is a class of Kripke n-frames and $\mathfrak{F}$ is a Kripke frame, then

- $L(\mathscr{C})$ denotes the set of $\mathcal{L}_n$-formulas valid on $\mathscr{C}$;
- $L_c(\mathscr{C})$ denotes the set of $\mathcal{L}_n$-formulas valid on every locally constant augmented n-frame over a Kripke frame from $\mathscr{C}$;
- $L^{=}(\mathscr{C})$ denotes the set of $\mathcal{L}_n^{=}$-formulas valid on $\mathscr{C}$;
- $L_c^{=}(\mathscr{C})$ denotes the set of $\mathcal{L}_n^{=}$-formulas valid on every locally constant augmented n-frame with equality over a Kripke frame from $\mathscr{C}$.

We write $L^{=}(\mathfrak{F})$ and $L_c^{=}(\mathfrak{F})$ rather than $L^{=}(\{\mathfrak{F}\})$ and $L_c^{=}(\{\mathfrak{F}\})$, respectively.

A Kripke n-frame $\langle W, R_1, \ldots, R_n\rangle$ is finite if W is a finite set. If L is an n-modal predicate logic (with or without equality), then L^{wfin} denotes the set of formulas valid on every finite Kripke frame validating L; this set is a normal n-modal predicate logic.

3. Main results

The following is our main technical result:

Proposition 1. *Let $\mathfrak{F}$ be a finite Kripke frame. Then, the monadic fragments with equality of the logics $L^{=}(\mathfrak{F})$ and $L_c^{=}(\mathfrak{F})$ are both decidable.*

From Proposition 1 we obtain the following:

Theorem 2. *Let $\mathscr{C}$ be a recursively enumerable class of finite Kripke n-frames. Then the monadic fragments with equality of the logics $L^{=}(\mathscr{C})$ and $L_c^{=}(\mathscr{C})$ are both in Π_1^0.*

It is known [12, Theorem 3.9] that, if L is a logic from one of the intervals $[\mathbf{QK}^{wfin}, \mathbf{QGL.3.bf}^{wfin}]$, $[\mathbf{QK}^{wfin}, \mathbf{QGrz.3.bf}^{wfin}]$ or $[\mathbf{QK}^{wfin}, \mathbf{QS5}^{wfin}]$, then the monadic fragment of L is Π_1^0-hard. Hence, Theorem 2, together with the transfer of completeness theorem for fusions [2, Theorem 4.1], give us the following:

Corollary 3. *Let $L = L_1 * \ldots * L_n$, where $L_1, \ldots, L_n$ are normal 1-modal propositional logics, such that*

- *$L_i \subseteq$ **S5** or $L_i \subseteq$ **GL.3** or $L_i \subseteq$ **Grz.3**, for some $i \in \{1, \ldots, n\}$;*
- *the class of finite Kripke frames validating L is recursively enumerable.*

Then, the monadic fragments of the logics $\mathbf{Q}L^{wfin}$*,* $\mathbf{Q}L.\mathbf{bf}^{wfin}$ *and the monadic fragments with equality of the logics* $\mathbf{Q}^{=}L^{wfin}$*,* $\mathbf{Q}^{=}L.\mathbf{bf}^{wfin}$ *are all* Π^0_1*-complete.*

Corollary 4. *Let* L *be one of the logics* **K**, **T**, **D**, **K4**, **K4.3**, **S4**, **S4.3**, **GL**, **GL.3**, **Grz**, **Grz.3**, **KB**, **KTB**, **K5**, **K45**, **S5**. *Then, the monadic fragments of the logics* $\mathbf{Q}L_n^{wfin}$*,* $\mathbf{Q}L_n.\mathbf{bf}^{wfin}$ *and the monadic fragments with equality of the logics* $\mathbf{Q}^{=}L_n^{wfin}$*,* $\mathbf{Q}^{=}L_n.\mathbf{bf}^{wfin}$ *are* Π^0_1*-complete.*

From Proposition 1 we also obtain the following:

Theorem 5. *Let* $\mathscr{C}$ *be a decidable class of Kripke* 1*-frames closed under the operation of taking subframes and satisfying the condition that there exists* $m \in \mathbb{N}$ *such that* $|R(w)| \leqslant m$ *whenever* $\langle W, R\rangle \in \mathscr{C}$ *and* $w \in W$*. Then the monadic fragments of the logics* $L(\mathscr{C})$*,* $L_c(\mathscr{C})$ *and the monadic fragments with equality of the logics* $L^{=}(\mathscr{C})$*,* $L^{=}_c(\mathscr{C})$ *are decidable.*

Recall that $\mathbf{Alt}_n$ is a monomodal logic complete with respect to the class of Kripke frames where every world sees at most n worlds. Using completeness of the predicate counterpart of $\mathbf{Alt}_n$ [16], which, using [3, Theorem 3.8.7], implies the completeness of $\mathbf{Q}^{=}\mathbf{Alt}_n$, we obtain the following:

Theorem 6. *The monadic fragments of logics* $\mathbf{QAlt}_n$*,* $\mathbf{QAlt}_n.\mathbf{bf}$*,* $\mathbf{Q}^{=}\mathbf{Alt}_n$*, and* $\mathbf{Q}^{=}\mathbf{Alt}_n.\mathbf{bf}$ *are all decidable.*

4. Discussion

Proposition 1 and Theorem 2 remain true in the Kripke semantics with equality as identity. Proposition 1 and Theorem 2 can be extended to logics of frames with distinguished worlds. All the results remain true if logics $\mathbf{Q}L.\mathbf{bf}$ and $\mathbf{Q}^{=}L.\mathbf{bf}$ are replaced with logics in which, for some k, the Barcan formula $\boldsymbol{bf}_k$ is replaced with $\Box\boldsymbol{bf}_k$, where $\Box$ is a finite sequence of $\Box_1, \ldots, \Box_n$. Lastly, we note that similar results can be obtained for superintuitionistic monadic logics [15].

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References

[1] Egon Börger, Erich Grädel, and Yuri Gurevich. *The Classical Decision Problem*. Springer, 1997.
[2] Dov Gabbay, Agi Kurucz, Frank Wolter and Michael Zakharyaschev. *Many-Dimensional Modal Logics*, volume 48 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2003.
[3] Dov Gabbay, Valentin Shehtman, and Dmitrij Skvortsov. *Quantification in Nonclassical Logic, Volume 1*, volume 153 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2009.
[4] Saul A. Kripke. The undecidability of monadic modal quantification theory. *Zeitschrift für Matematische Logik und Grundlagen der Mathematik*, 8:113–116, 1962.
[5] Mikhail Rybakov. Undecidability of modal logics of unary predicate. *Logical Investigations*, 23(2):60–75, 2017. (In Russian)
[6] Mikhail Rybakov. Computational complexity of binary predicate theories with a small number of variables in the language. *Doklady Mathematics*, 507(6):61–65, 2022.
[7] Mikhail Rybakov. Binary predicate, transitive closure, two-three variables: shall we play dominoes? To appear in *Logical Investigations*. (In Russian)
[8] Mikhail Rybakov. Predicate counterparts of modal logics of provability: High undecidability and Kripke incompleteness. To appear in *Logic Journal of the IGPL*.
[9] Mikhail Rybakov and Dmitry Shkatov. Undecidability of first-order modal and intuitionistic logics with two variables and one monadic predicate letter. *Studia Logica*, 107(4):695–717, 2019.
[10] Mikhail Rybakov and Dmitry Shkatov. Trakhtenbrot theorem for classical languages with three individual variables. *Proceedings of the South African Institute of Computer Scientists and Information Technologists 2019*, Skukuza, South Africa, September 17–18, 2019, ACM, NY, USA, Article No. 19:1–7. 2019.
[11] Mikhail Rybakov and Dmitry Shkatov. Algorithmic properties of first-order modal logics of the natural number line in restricted languages. In Nicola Olivetti, Rineke Verbrugge, Sara Negri, and Gabriel Sandu, editors, *Advances in Modal Logic*, volume 13. College Publications, 2020.
[12] Mikhail Rybakov and Dmitry Shkatov. Algorithmic properties of first-order modal logics of finite Kripke frames in restricted languages. *Journal of Logic and Computation*, 30(7):1305–1329, 2020.
[13] Mikhail Rybakov and Dmitry Shkatov. Algorithmic properties of first-order superintuitionistic logics of finite Kripke frames in restricted languages. *Journal of Logic and Computation*, 31(2):494–522, 2021.
[14] Mikhail Rybakov and Dmitry Shkatov. Algorithmic properties of first-order modal logics of linear Kripke frames in restricted languages. *Journal of Logic and Computation*, 31(5):1266–1288, 2021.
[15] Mikhail Rybakov and Dmitry Shkatov. Algorithmic properties of modal and superintuitionistic logics of monadic predicates over finite frames. *Submitted to the Journal of Logic and Computation.*
[16] Valentin Shehtman and Dmitry Shkatov. Some prospects for semiproducts and products of modal logics. In N. Olivetti and R. Verbrugge, editors, *Short Papers Advances in Modal Logic AiML 2020*, pages 107–111. University of Helsinki, 2020.

[17] Alfred Tarski and Steven Givant. *A Formalization of Set Theory without Variables*, volume 41 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, 1987.
[18] Frank Wolter and Michael Zakharyaschev. Decidable fragments of first-order modal logics. *The Journal of Symbolic Logic*, 66:1415–1438, 2001.

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DYNAMIC EPISTEMIC LOGIC FOR BUDGET-CONSTRAINED AGENTS

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This abstract is based on recent work in [6]. We present a static ($\mathsf{EL_{bc}}$) and dynamic ($\mathsf{DEL_{bc}}$) epistemic logic for budget-constrained agents. $\mathsf{EL_{bc}}$ extends a standard multi-agent epistemic logic with expressions concerning agent's budgets and formulas' costs. $\mathsf{DEL_{bc}}$ extends $\mathsf{EL_{bc}}$ with dynamic modality "$[?_iA]\varphi$" which reads as "φ holds after i's question whether a propositional formula A is true". We provide a sound and complete axiomatization for $\mathsf{EL_{bc}}$ and $\mathsf{DEL_{bc}}$ and show that both logics are decidable.

1. Motivation

Dynamic epistemic logic [18, 11, 14] is a common way of describing agents' knowledge and informational changes. But nowadays, our intuition about the nature of agents' reasoning and interaction tells us that both processes of operating with available knowledge and obtaining a new one cannot always be effortless. This natural intuition demonstrates that reasoning often becomes a resource consuming action. A lot of researchers of epistemic logic paid attention to this problem and found different approaches to formalising the idea of resource-bounded agents [9]. The wide range of existing approaches, describing non-omniscient agents, consider resources as various cognitive limits.

Non-omniscience can be described through time- and memory-constrained agents who do not necessarily know all the logical consequences of their knowledge. Some papers model such constraints through so-called inferential actions, which require agents to take explicit inference steps, spending available resources to deduce the logical consequences of their knowledge [16]. Other papers extend the idea of a bounded deliberation process with resource consuming inference actions by introducing perception [4] or rule-based models [13] and their effects on formation of agents' beliefs. The idea of resource-bounded agents, situated in agent–environment systems that takes into account agents' observations, beliefs, goals and actions, sounds promising both for philosophers and computer scientists [2]. Most contemporary papers on resource-bounded reasoning would agree that modelling of non-omniscient agents does not mean modelling of imperfect reasoners. On the contrary, a lot of papers argue that epistemic logic must formalise the idea that if the agent knows all necessary premises and either thinks hard enough [7] or has enough time [3, 1], then they will know the conclusion. Thus, the reasoning process itself can justifiably be considered as an ongoing time-consuming [12], as well as a memory-consuming [8] process. This intuition bridges the gap between reasoning and computation process and sounds fruitful for AI research. While this is a reasonable assumption which is worth studying, both time- and memory-based approaches deal with 'inner' obstacles of an agent's deliberation process. Thus, even existing papers studying resource constraints in agent-environment settings consider resources as a tool of reasoning or obtaining new information from already available agent's knowledge. At the same time, a lot of real-life scenarios demonstrate that resources can also be considered as an instrument of obtaining new, independent or already available, information from the outside. In other words, solving some tasks can require getting additional information, which is not necessarily costless. Our main goal is to consider logically omniscient reasoners who can interact with the environment (in the sense of an independent bystander) and obtain new information from this environment by spending a certain amount of resources.

A similar attempt was made by Naumov and Tao [15]. Their paper describes budget-constrained agents in epistemic settings. It catches the intuition that sometimes agents have to spend their resources to obtain the knowledge of some fact. But since their logic is static and describes resource constraints as a feature of the knowledge operator itself, this approach violates the Negative Introspection axiom, so it appears to be a S4-like system. Nevertheless, this S4-like epistemic logic is complete, with respect to S5-like structures. We aims to demonstrate that reasoning about knowledge and informational change under budget constraints can be described by an S5-like system if we consider this informational change explicitly in DEL-style language.

We assume that agents can purchase information, spending some resources available to them. Intuitively, agents can ask a question "is A true?" and get a positive or negative answer. Sometimes, this question can require some resources.

2. Language

Let $Prop = \{p, q, \dots\}$ be a countable set of propositional letters. Denote by PL the set of all propositional (non-modal) formulas defined by the following grammar (where p ranges over $Prop$, other connectives are defined in a standart way): $A, B \ ::= \ p \ | \ \neg A \ | \ (A \wedge B)$.

The language $\mathsf{EL_{bc}}$. Let $\mathsf{Agt} = \{i, j, \dots\}$ be a finite set of agents. We fix a set of constants $Const = \{c_A \mid A \in PL\} \cup \{b_i \mid i \in \mathsf{Agt}\}$. It contains a constant c_A for the *cost* of each propositional formula A and a constant b_i for the *budget* of each agent i. *Formulas* of the language $\mathsf{EL_{bc}}$ are defined by the following grammar:

$$\varphi, \psi \ ::= \ p \mid (z_1 t_1 + \ldots + z_n t_n) \geq z \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_i \varphi$$

where p ranges over Prop, $i \in \mathsf{Agt}$, $t_1, \dots, t_n \in \mathsf{Const}$ and $z_1, \dots, z_n, z \in \mathbb{Z}$.

Other Boolean connectives $\to, \vee, \leftrightarrow, \bot$ and $\top$ are defined in the standard way. The dual operator for K_i is defined as $\hat{K}_i\varphi := \neg K_i \neg\varphi$. We will also use $K_i^? \varphi$ as an abbreviation for $(K_i\varphi \vee K_i \neg\varphi)$. Note that we introduce the cost c_A only for propositional formulas $A \in PL$. The logic with costs of arbitrary epistemic formulas is left for future research. We deal with linear inequalities and use the same abbreviations as in [10]. Thus, we write $t_1 - t_2 \geq z$ for $t_1 + (-1)t_2 \geq z$, $t_1 \geq t_2$ for $t_1 - t_2 \geq 0$, $t_1 \leq z$ for $-t_1 \geq -z$, $t_1 < z$ for $\neg(t_1 \geq z)$, and $t_1 = z$ for $(t_1 \geq z) \wedge (t_1 \leq z)$. Thus, the language $\mathsf{EL_{bc}}$ allows us to express statements such as: "$c_{p\wedge q} \geq 7$", "$b_i \geq 5$", "$2b_i = b_j$", "$K_c(b_i + b_j \geq c_{p \vee q})$" etc.

The set of subformulas $\mathsf{Sub}(\varphi)$ of a formula φ is defined in the standard way note that if a constant c_A occurs in φ then we do not count A as a subformula of φ.

3. Semantics

A *model* $\mathcal{M}$ of the logic $\mathsf{EL_{bc}}$ has the components standard for the multi-modal logic **S5**, namely, a non-empty set of states W, an epistemic accessibility relation $\sim_i$ for each agent $i \in \mathsf{Agt}$, and a valuation $V\colon \mathsf{Prop} \to 2^W$. Besides, a model $\mathcal{M}$ contains a function Cost that assigns to every propositional formula at each state its cost, and a function Bdg that assigns to each agent $i \in \mathsf{Agt}$ at each state $w \in W$ the available amount of resources.

A *model* is a tuple $\mathcal{M} = (W, (\sim_i)_{i \in \mathsf{Agt}}, \mathsf{Cost}, \mathsf{Bdg}, V)$, where

- W is a non-empty set of *states*,
- $\sim_i \subseteq (W \times W)$ is an equivalence relation for each $i \in Agt$,
- $Cost\colon W \times PL \longrightarrow \mathbb{R}^+$ is the (non-negative) *cost* of propositional formulas,
- $Bdg\colon Agt \times W \longrightarrow \mathbb{R}^+$ is the (non-negative) *bugdet* of each agent at each state,
- $V : \mathsf{Prop} \to 2^W$ is a *valuation* of propositional variables.

Thus both the cost of a formula and the budget of an agent depend on a current state. We use $\mathsf{Bdg}_i(w)$ as an abbreviation for $\mathsf{Bdg}(i, w)$, where $i \in \mathsf{Agt}$ and $w \in W$. In order to formulate additional constraints on the function Cost, we need the following notation. Let PL be the classical propositional logic. For any propositional formulas A and B:

- A and B are called *equivalent*: $A \equiv B$ iff $\vdash_{\mathrm{PL}} A \leftrightarrow B$,
- A and B are called *similar*: $A \approx B$ iff $A \equiv B$ or $A \equiv \neg B$.

We also impose the following conditions on the function Cost:

(C1) $\mathsf{Cost}(w, \bot) = \mathsf{Cost}(w, \top) = 0$,
(C2) $A \approx B$ implies $\mathsf{Cost}(w, A) = \mathsf{Cost}(w, B)$, for all $A, B \in PL$ and all $w \in W$.

The *truth* $\vDash$ of a formula A at a state $w \in W$ of a model $\mathcal{M}$ is defined by induction:$\mathcal{M}, w \vDash p$ iff $w \in V(p)$,
$\mathcal{M}, w \vDash \neg\varphi$ iff $\mathcal{M}, w \nvDash \varphi$,
$\mathcal{M}, w \vDash \varphi \wedge \psi$ iff $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$,
$\mathcal{M}, w \vDash K_i\varphi$ iff $\forall w' \in W\colon w \sim_i w' \Rightarrow \mathcal{M}, w' \vDash \varphi$,
$\mathcal{M}, w \vDash (z_1 t_1 + \cdots + z_n t_n) \geq z$ iff $(z_1 t'_1 + \cdots + z_n t'_n) \geq z$, where for $1 \leq k \leq n$,

$$t'_k = \begin{cases} \mathsf{Cost}(w, A), & \text{for } t_k = c_A, \\ \mathsf{Bdg}_i(w), & \text{for } t_k = b_i. \end{cases}$$

We refer to the class of all models satisfying all properties mentioned above as $\mathfrak{M}$. We write $\vDash_{\mathfrak{M}} \varphi$ if the formula φ is *valid* in the class of models $\mathfrak{M}$.

4. Axiomatization

The axiomatisation of the logic $\mathsf{EL_{bc}}$ is presented in Table 1. Here, (Ineq) is the set of axioms for linear inequalities firstly described in [10] and used later for similar purposes in [16].

Table 1. Proof system for $\mathsf{EL_{bc}}$

	Axioms:
(Taut)	All instances of propositional tautologies
(Ineq)	All instances of the axioms for linear inequalities
(K)	$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
(T)	$K_i\varphi \to \varphi$
(4)	$K_i\varphi \to K_iK_i\varphi$
(5)	$\neg K_i\varphi \to K_i\neg K_i\varphi$
(Bd)	$b_i \geq 0$
($\geq_1$)	$c_A \geq 0$
($\geq_2$)	$c_\top = 0$
($\geq_3$)	$c_A = c_B$ if $A \approx B$, for all formulas $A, B \in PL$
	Inference rules:
(MP)	From φ and $\varphi \to \psi$, infer ψ
(Nec$_i$)	From φ infer $K_i\varphi$

Axiomatization of $\mathsf{DEL_{bc}}$ can be obtained by adding the reduction axioms from Table 2 to the axiomatization of $\mathsf{EL_{bc}}$. The notation $\big((z_1t_1 + \cdots + z_nt_n) \geq z)\big)^{[b_i \backslash (b_i - c_A)]}$ means that all occurrences of b_i in $(z_1t_1 + \cdots + z_nt_n) \geq z$ are replaced with $(b_i - c_A)$.

Table 2. Reduction axioms and inference rules

(R_p)	$[?_iA]p \leftrightarrow (b_i \geq c_A) \to p$
$(R_\geq)$	$[?_iA]\big((z_1t_1 + \cdots + z_nt_n) \geq z)\big) \leftrightarrow (b_i \geq c_A) \to$ $\to \big((z_1t_1 + \cdots + z_nt_n) \geq z)\big)^{[b_i \backslash (b_i - c_A)]}$
$(R_\neg)$	$[?_iA]\neg\varphi \leftrightarrow (b_i \geq c_A) \to \neg[?_iA]\varphi$
$(R_\wedge)$	$[?_iA](\varphi \wedge \psi) \leftrightarrow [?_iA]\varphi \wedge [?_iA]\psi$
(R_{K_j})	$[?_iA]K_j\varphi \leftrightarrow (b_i \geq c_A) \to K_j[?_iA]\varphi$, where $i \neq j$
(R_{K_i})	$[?_iA]K_i\varphi \leftrightarrow (b_i \geq c_A) \to$ $\to \Big(\big(A \to K_i(A \to [?_iA]\varphi)\big) \wedge \big(\neg A \to K_i(\neg A \to [?_iA]\varphi)\big)\Big)$
(Rep)	From $\vdash \varphi \leftrightarrow \psi$, infer $\vdash [?_iA]\varphi \leftrightarrow [?_iA]\psi$

Theorem 1. *The logic $\mathsf{EL_{bc}}$ is sound and complete w.r.t. $\mathfrak{M}$.*

Here we prove only weak completeness result due to non-compactness of $\mathsf{EL_{bc}}$. To see that $\mathsf{EL_{bc}}$ is non-compact consider a set of $\mathsf{EL_{bc}}$-formulas: $\{c_A > n \mid n \in \mathbb{N}\}$. It is easy to see that any finite subset of this set is satisfiable while the set itself is not.

Theorem 2. *The logic $\mathsf{DEL_{bc}}$ is sound and weakly complete w.r.t. $\mathfrak{M}$.*

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References

[1] N. Alechina, B. Logan. Ascribing Beliefs to Resource Bounded Agents. *Proceedings of the First International Joint Conference on Autonomous Agents and Multiagent Systems*, 881–888, 2002.

[2] N. Alechina, B. Logan. A Logic of Situated Resource-Bounded Agents. *Journal of Logic, Language and Information*, 18, 79–95, 2009.

[3] N. Alechina, B. Logan. M. Whitsey. A Complete and Decidable Logic for Resource-Bounded Agents. *Autonomous Agents and Multi-Agent Systems*, 606–613, 2004.

[4] P. Balbiani, D. Fernandez-Duque, E. Lorini The Dynamics of Epistemic Attitudes in Resource-Bounded Agents. *Studia Logica*, 107, 457–488, 2019.

[5] S. Dautović, D. Doder, Z. Ognjanović. An Epistemic Probabilistic Logic with Conditional Probabilities. *Logics in Artificial Intelligence*, 279–293, 2021.

[6] V. Dolgorukov, M. Gladyshev. Dynamic Epistemic Logic for Budget-Constrained Agents, *Dynamic Logic. New Trends and Applications*, 56–72, 2023.

[7] H. Duc. Reasoning about Rational, but not Logically Omniscient, Agents. *Journal of Logic and Computation*, 7(5), 633–648, 1997.

[8] J. Elgot-Drapkin, M. Miller, D. Perlis, Memory, Reason and Time: the Steplogic Approach, *Philosophy and AI: Essays at the Interface*, 1991.

[9] R. Fagin, J. Halpern. Belief, Awareness, and Limited Reasoning. *Artificial Intelligence*, 34, 39–76, 1988.

[10] R. Fagin, J. Halpern. N. Megiddo. A Logic for Reasoning about Probabilities. *Information and Computation*, 87 (1), 78–128, 1990.

[11] R. Fagin, J. Halpern, Y. Moses, M. Vardi. *Reasoning about knowledge*, 1995.

[12] J. Grant, S. Kraus, D. Perlis. A Logic for Characterizing Multiple Bounded Agents. *Autonomous Agents and Multi-Agent Systems*, 3(4), 351–387, 2000.

[13] M. Jago. Epistemic Logic for Rule-Based Agents. *Journal of Logic, Language and Information*, 18(1), 131–158, 2009.

[14] B. Kooi. Dynamic Epistemic Logic. *Handbook of Logic and Language*, 671–690, 2011.

[15] P. Naumov, J. Tao. Budget-constrained Knowledge in Multiagent Systems. *Autonomous Agents and Multi-Agent Systems*, 219–226, 2015.

[16] A. Solaki. Bounded Multi-agent Reasoning: Actualizing Distributed Knowledge *DaLi 2020: Dynamic Logic. New Trends and Applications 2020*, 239–258, 2020.

[17] H. van Ditmarsch, J. Fan . Propositional Quantification in Logics of Contingency. *Journal of Applied Non-Classical Logics*, 26 (1), 81–102, 2016.

[18] H. van Ditmarsch, W. van der Hoek, B. Kooi. *Dynamic Epistemic Logic*, 2007.

[19] Y. Wang, Q. Cao. On Axiomatizations of Public Announcement Logic. *Synthese*, 190, 103–134, 2013

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THE STRUCTURAL DEFINITION OF LOGICAL NEGATION THROUGH THE DOUBLY NEGATED PROPOSITIONS

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In the second half of 20$^{\text{th}}$ Century scholars of mathematical logic privileged the double negation law with respect to the excluded middle law in the role of differentiating classical logic from most kinds of non-classical logic, first of all intuitionist logic, and then the minimal one, the positive one, etc. (Prawitz and Melmnaas 1968; Prawitz 1976).

Against scientists' custom of introducing into both Logic and Mathematics idealistic notions, respectively Intuitionists (Brouwer 1975, Dummett 1977) and Constructivists (Bishop 1967, pp. 1–10) claimed a basic requirement, i.e. a proposition is true only if is supported by factual evidence; in the case a proposition does not satisfy such a requirement and there is no evidence of its falsity, it has to be replaced by its doubly negated proposition according to the "[doubly] negative translation" (Troelstra and van Dalen 1988, pp. 56ff.); one obtains a doubly negated proposition which is not equivalent to the corresponding affirmative proposition owing to latter's lack of evidence (DNP). In the following each doubly negated proposition whose corresponding affirmative one lacks of evidence or even is different in meaning will be considered a DNP pertaining to intuitionist logic.

Along two millennia a clear presentation of a scientific theory was equated to a systematic theoretical organization of a deductive kind from few axioms, irrespectively they were supported by evidence or not (AO). It is easy to recognize that this kind of organization does not include a DNP because it is impossible to draw a DNP from an axiom, which, being an affirmative proposition is equivalent to its doubly negated proposition. On the other hand, a DNP cannot play the role of an axiom of an axiomatic system, because its content cannot be defined with certainty, except for stating a bound to our thinking; and in any case, no affirmative proposition can be derived from a DNP, for which the double negated law fails. The question arises of how DNPs may compose a theory.

In the past, several scientific theories have been presented by their respective authors in a non-axiomatic way. A comparative analysis on their original texts shows the characteristic features of a common model of theoretical organization.

i) No more than the common knowledge on the field at issue is presupposed.

ii) A problem which is unsolvable through usual tools is declared; e.g., in Lavoisier's 18th century chemistry: what are the elements of the matter; in Lazare Carnot's mechanics: what are the invariants in a collision of bodies; in S. Carnot's thermodynamics: what is the maximum efficiency of the heat/work transformations; in Lobachevsky's geometry: how many parallel line exist; in Kolmogorov's 1924/25 paper: why has the illegitimacy of the use of the law of excluded middle gone unnoticed. I call this problem-based kind of theoretical organization a PO.

iii) The aim is to invent a new scientific method capable to solve the given problem.

iv) The theory argues by linking together the DNPs in order to constitute — through words as "otherwise we obtain an absurd result" — an ad absurdum argument (AAA); which is of a weak kind, i.e. its final proposition is a DNP (a common opinion wrongly holds that an ad absurdum argument concludes an affirmative proposition; that is true only if one applies to its conclusion the double negation law, which yet pertains to classical logic). This DNP may work as a premise for a next AAA; these AAAs constitute a chain of AAAs.

v) The conclusion of the final AAA is a universal predicate, $\neg\neg UP$, which represents a possible resolution for all cases of the stated problem.

vi) At this point, the author of such kind a theory as a matter of fact translates the above predicate UP into the corresponding affirmative predicate UP. One can suppose that the author thinks to have already collected enough reasoning evidence to be justified in promoting his conclusion $\neg\neg UT$ to the corresponding affirmative proposition UP, although this change is not allowed by intuitionist logic, which previously he had adhered to.

vii) After this step the author considers the affirmative proposition UP as a valid hypothesis for resolving the given problem; i.e. he assumes this proposition as a new axiom, from which, precisely because it is affirmative, he can draw by means of classical logic all possible derivations, to be subsequently tested against reality (Drago 2012).

viii) Previous step apparently represents an application of Leibniz' principle of sufficient reason; whose antecedent is itself a universal DNP ("Nothing is without reason") and the consequent is the corresponding affirmative proposition ("Everything has a reason"). An author of a theory performs on a specific predicate of his theory this logical step which formally is the same step performed by the application of this general principle, i.e. the logical step occurring from the antecedent of Leibniz' principle of sufficient reason to its consequent: $\neg\exists x\, \neg P(x) \Rightarrow \forall x\, P(x)$. In formal terms of the square of opposition PSR, this step translated into the classical thesis A ("S is P") the intuitionist version of the main thesis, A^{I} ("not (S is not P)"). Through Dummett's table (Dummett 1977, p. 29) it is apparent that previous translation is enough for changing the entire intuitionist predicate logic into the classical one (e.g. the similar change of thesis E^{I} is obtained by mere negation of thesis A^{I}; the change of thesis I^{I} is obtained by doubly negating thesis A^{I}).

Notice that the translation performed by PSR merely constitutes the inverse translation of the so-called 'negative translation'. Yet, whereas the latter one can be always performed under some rules about the suitable way to add the two negations to a predicate, the application of PSR is valid, as Markov stated, under two requirements on the doubly negated predicate on which it is applied: it has to be 1) derived from an AAA and 2) decidable (Markov 1961, p. 5; Drago 2012, sect. 7) for the simple reason that for passing from a hypothetical world to a real world an author has to be supported not only by logical arguments, but also reality criteria.

In sum, the sense of a PO theory is the following one: the conclusion of the final AAA suggests a surmise $\neg\neg UP$; but a surmise, being a DNP, cannot be accurately tested with reality; instead, the application of PSR translates this conclusion into an affirmation which may be tested. This step also changes the theoretical development, previously based upon intuitionist logic, into a deductive development based upon classical logic in order to test with reality all the possible derivations from the new principle. If the obtained answers are positive, the entire theory becomes effective. In sum, what was hypothetical is changed into a suggestion on the real world.

In sum, from the above listed scientific theories playing prominent roles within the history of science I conclude that there exists a new model of a theoretical organization, which is governed by intuitionist logic and has to be put on the same par of AO. Hence, a DNP severs two entirely different (scientific) worlds, i.e. a world of assured truths, as those drawn from assured principles-axioms, from a world of inductive searching for discovering a new method for solving basic problems. (Drago 2012)

As a consequence, even the definition of negation has to be referred to the theoretical organization which it belongs to. Within an AO governed by classical logic all doubly negated propositions are exactly equal to the corresponding affirmative propositions and hence a negative one is mirror opposite to the corresponding affirmative one; instead, within a PO, governed by intuitionist logic, only the DNP is not false, whereas both negation or affirmation are at most partially true. Therefore, the common basic notion of negation presents within the two kinds of logic a radical variation in its value and hence also in its meaning. This variation gives reason why scholars wanting to define in general terms the meaning of a negation by only considering the logical operations inside a specific kind of logic (usually, classical logic) met insurmountable difficulties. (Horn and Wansing 2015)

But also inside intuitionist logic it is difficult to definition negation, so that this question represents the main basic problem of its philosophy. Brouwer suggested that a negative proposition implies, through a specific constructive proof, the absurdity. "Brouwer [papers 1923a–c]... expressed negation as reasoning that leads to an absurdity, or, briefly, as an absurdity." (Franchella 1994, p. 258. The book (de Stigt 1990, pp. 238-270) offers a more detailed analysis of the subject). Kolmogorov (1924/25, §§3–6, pp. 420–422) objected that this formalization implies to have obtained in a preliminary way the notion of negation either in the law of contradiction or the AAA. After a century, the long debate on this question is still unresolved (Sundholm 1994; Raatikainen 2004; Raatikainen 2013). The latter author distinguished three attitudes; *i)* strict actualism, i.e., to assume the existence of the contradiction proof in all cases; *ii)* possibilism, i.e. the proof is no more than possible and *iii)* an intermediated case of liberalized actualism. He concluded as follows: "We have examined the three basic choices there are for the intuitionist theory of truth, the strict actualism, the liberalized actualism and possibilism, and found all them wanting." (Raatikainen 2004, p. 143)

I suggest that the question is still unresolved because in the past it was scrutinized with reference to only AO. By taking into account also PO, the differences about the relationship between intuitionist negation and a proof of absurdity is clear. The intuitionist logic of the kind AO, i.e. Heyting's axiomatic logic, considers a negation as suggesting the existence of a proof of its absurdity no matter of the idealistic nature of this claimed existence; whereas (Kolmogorov 1932) formulated logic according to PO, which in general suggests no more than hypotheses (Drago 2021); hence, about a negation it suggests a mere hypothesis of the existence of a proof of its absurdity; that is, in a PO theory not always this proof actually exists. So we have two methods obtaining a same notion, intuitionist negation, but attributing to it two different meanings. According to the first method the proof may be considered as idealistically actual; in the second case the proof is no more than possible. In this latter case one achieves an affirmative conclusion of the theory by changing the kind of theoretical organization by means of the application of the PSR, i.e. an appeal through to the human rationality. The same occurs about the existence of the proof related to a negation before the PO of a theory is changed by PSR into a deductive development: this proof is merely possible; and in order to state a negation one has to exhibit a constructive proof of its absurdity; instead, after the change, when the theory is governed by classical logic, the proof is always assumed as actual, albeit also in an idealistic sense.

In conclusion, the nature of the definition of negation is not of an objective kind, or worst, as in Brouwer's sense, an absurd of subjective nature, rather it is of a structural kind. It depends from the kind of theoretical organization which it belongs to.

Rightly Carnap (1943) distrusted the "logical inferentialism", i.e. the commonly shared opinion that all logical constants can be uniquely defined through a local analysis of inference relations. Moreover, in order to give an exact meaning to a negation Dummett's interpretation, called "meaning-as-use", has to be extended beyond the use of the single inferences, till up to the use of the two theoretical organizations.

Finally I suggest that, since the problem of what is a negation plays a crucial role inside the theory of intuitionist logic, one may re-formulate this logic according to a PO whose basic problem is this one.

References

[1] Brouwer L.E.J. (1975), *Collected Works*, Amsterdam: North-Holland.
[2] Carnap, R. (1943). *Formalization of logic.* Cambridge, MA: Harvard University Press
[3] de Stigt W.P. (1990), *Brouwer's Intuitionism,* Amsterdam: North-Holland.
[4] Drago A. (2012), "Pluralism in Logic. The Square of opposition, Leibniz's principle and Markov's principle", in *Around and Beyond the Square of Opposition,* edited by J.-Y. Béziau and D. Jacquette, Basel: Birckhaeuser, 175–189.
[5] Drago A. (2021), "An Intuitionist Reasoning upon Formal Intuitionist Logic: Logical Analysis of Kolmogorov's 1932 Paper", *Logica Universalis,* 15(4), 2021, pp. 537–552.
[6] Dummett M. (1977), *Elements of Intuitionism,* Oxford: Oxford U.P..
[7] Franchella M. (1994), "Brouwer and Griss on intuitionistic negation", *Modern Logic,* 4, pp. 256–265.
[8] Horn L. and Wansing H. (2015), "Negation" in Zalta E.N., (ed.) *Stanford Encyclopedia of Philosophy.* Stanford, https://plato.stanford.edu/entries/negation/
[9] Kolmogorov A.N. (1924/25), "On the principle of "tertium non datur", *Mathematicheskii Sbornik,* 32 (1924/25) 646–667 (Engl. tr. In van Heijenoorth, pp. 416–437).
[10] Kolmogorov A.N. (1932), "Zur Deutung der Intuitionistischen Logik", *Math. Zeitfr.*, 35, 58–65 (Engl. Transl. In Mancosu (1998), pp. 328–334).
[11] Markov A.A. (1961), "On constructive mathematics". *Trudy Mathematichieskie Institut Steklov,* 67, pp. 8–14 (English translation: 1971, *Am. Math. Soc. Trans.* 98(2), pp. 1–9).
[12] Prawitz D. (1977), "Meaning and proofs: on the conflict between classical and intuitionistic logic", *Theoria,* 43 (1), pp. 2–40.
[13] Prawitz D., Melmnaas P.-E. (1968), "A survey of some connections between classical intuitionistic and minimal logic". In Schmidt H.A., Schuette K., Thiele E.-J. (eds.), *Contributions to Mathematical Logic,* Amsterdam: North-Holland, 215–230.
[14] Raatikainen P. (2004), "Conceptions of truth in intuitionism", *History and Philosophy of Logic,* 25, pp. 131–145.
[15] Raatikainen P. (2013), "Intuitionist logic and its philosophy", *Al Mukhtabat,* 6, pp. 114–127.
[16] Sundholm G. (1994), "Existence, proof and truth-making: A perspective on the intuitionist conception of truth", *Topoi,* 13, pp. 117–126.
[17] Troelstra A. and van Dalen D. (1988), *Constructivism in Mathematics,* Amsterdam: Norh-Holland.

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CONSERVATIVE TRANSLATIONS FOR NON-DETERMINISTIC SEMANTICS

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In this work, joint with Pawel Pawlowski, we prove that a certain subclass of non-deterministic semantics for two, three and four-valued propositional logic may be conservatively translated to modal logic. We obtain this result straightforwardly adapting the results of [1] and [2] to fit the non-deterministic setting. We formulate our results in a more abstract format, in such a way to single out the sufficient properties for a (non-)deterministic n-valued propositional logic to be conservatively translated to a certain modal logic. Our result is a further step towards the understanding of the features of non-deterministic semantics. Many related questions are left open.

The notion of conservative translation is widely used in the literature, as a mean to compare the properties of two logics. We emphasise that a conservative translation preserves the validity of arguments between two logics, and not only the validity of formulae.

Definition 1 (Conservative translation). *A conservative translation between logics* $\mathtt{L_1}, \mathtt{L_2}$ *is a function* T *such that*

$$\Gamma \models_{\mathtt{L}_1} \varphi \Leftrightarrow T(\Gamma) \models_{\mathtt{L}_2} T(\varphi)$$

for any subset Γ *of formulae and any formula* φ *in the language of* $\mathtt{L_1}$.

Tamminga and Kooi in [1] provide a conservative translation for three-valued deterministic propositional logics into the modal logic **S5** with full semantics (see Definition 2). Kubyshkina in [2] generalised this result, providing a conservative translation for four-valued deterministic propositional logics into universal neighbourhood models (see Definition 3).

The set $\mathtt{Prop}$ denote the set of propositional variables. The modal language $\mathcal{ML}$ is the standard modal language with connectives $\mathtt{C} = \{\neg, \wedge, \vee, \Diamond, \Box\}$.

Definition 2 (**S5** full model). *An **S5** full model, or simply **S5** model,* $\mathcal{M}$ *is a pair* $\langle W, V \rangle$, *where* $W \neq \emptyset$ *is a set of worlds and* $V : \mathtt{Prop} \to \mathcal{P}(W)$ *is a valuation.*

The satisfaction relation $\models_{\mathbf{S5}}$ for **S5** full models is defined according to the standard Kripke semantics, assuming that each world is accessible to any other.

Definition 3 (universal neighbourhood model). *A triple* $\mathcal{M} = \langle W, N, V \rangle$ *is a universal neighbourhood model if and only if:*

- $W \neq \emptyset$,
- $N : W \to \mathcal{P}(W)$,
- $V : \mathtt{Prop} \to \mathcal{P}(W)$ *is a valuation,*
- *for each* $w, w' \in W$, $N(w) = N(w')$.

The satisfaction relation $\models_{\mathbf{N}}$ for universal neighbourhood models is defined as usual for the propositional connectives, so we limit to recall the clauses for the modal operators:

- $\mathcal{M}, w \models_{\mathbf{N}} \Diamond\varphi$ if and only if $\{w \in W \mid \mathcal{M}, w \not\models \varphi\} \notin N(w)$
- $\mathcal{M}, w \models_{\mathbf{N}} \Box\varphi$ if and only if $\{w \in W \mid \mathcal{M}, w \models \varphi\} \in N(w)$.

Despite their difference these two semantics for modal logic share two relevant properties, which are needed in our proof (and in those of Tamminga and Kooi, and Kubyshkina) of the existence of a conservative translation from certain non-deterministic propositional logics to certain modal logics. This fundamental properties are captured in the following definition.

Fully-modalised formulae $\mathcal{ML}^-$ are the formulae obtained closing under $\neg, \wedge, \vee$ the set $\{\Diamond p \mid p \in \mathtt{Prop}\} \cup \{\Box p \mid p \in \mathtt{Prop}\}$.

Definition 4. *A semantics* $\models_{\mathbf{Mod}}$ *for modal logic is k-good, for some* $k \in \mathbb{N}$, *if it satisfies the following conditions:*

(1) *the interpretation of the propositional connective is the standard one, given for example by the Kripke semantics,*

(2) *there exist* $\bigcirc_1 p, \ldots, \bigcirc_k p$ *fully modalised formulae whose unique propositional variable is* p *such that* $\forall p\, \exists! i \leq k\, \mathcal{M} \models_{\mathbf{Mod}} \bigcirc_i p$,
(3) *for every fully-modalised formula* φ, $\mathcal{M}, w \models_{\mathbf{Mod}} \varphi$, *for some world* w, *if and only if* $\mathcal{M} \models_{\mathbf{Mod}} \varphi$.

The main difference between a deterministic and a non-deterministic semantics is that in the latter the interpretation of a given connective does not uniquely determine the truth-value of the whole formula. So, in this context, the assignment of values to propositional parameters does not single out a valuation defined over the full language. On the opposite, one assignment can be extended into multiple valuations. Formally non-deterministic (many-valued) semantics are defined as follows.

Definition 5. *Let* $\mathcal{L}_{\mathtt{C}}$ *be a propositional language. A non-deterministic matrix* $\mathtt{M}$ *(n-matrix) for* $\mathcal{L}_{\mathtt{C}}$ *is a triple* $\langle \mathtt{Val}, \mathtt{D}, \mathtt{O}\rangle$, *where:*

- $\mathtt{Val}$ *is a non-empty set of truth-values.*
- $\mathtt{D} \neq \emptyset$, $\mathtt{D} \subseteq \mathtt{Val}$, *is the set of designated values.*
- $\mathtt{O}$ *is a set of functions, which for any n-ary connective* $\maltese_n$ *of* $\mathcal{L}_{\mathtt{C}}$ *contains a corresponding n-ary function* $f_{\maltese_n} : \mathtt{Val}^n \to 2^{\mathtt{Val}} \setminus \{\emptyset\}$.

We say that a n-matrix is k-valued if and only if $|\mathtt{Val}| = k$.

A valuations v *in* $\mathtt{M}$ *is a function* $v\colon Form_{\mathcal{L}_{\mathtt{C}}} \to \mathtt{Val}$ *such that, for any sequence of formulae* $\varphi_1, \ldots \varphi_n$, $v(\maltese_n(\varphi_1, \ldots, \varphi_n)) \in f_{\maltese_n}(v(\varphi_1), \ldots, v(\varphi_n))$. *A valuation* v *is called assignment when its domain is restricted to the propositional variables, that is* $v\colon \mathtt{Prop} \to \mathtt{Val}$. *A valuation* w *extends an assignment* v, *in symbols* $w \supseteq v$, *if and only if, for each* $p \in \mathtt{Prop}$, $w(p) = v(p)$.

The subclass of non-deterministic semantics for k-valued propositional logics we are interested in is captured by the following definition.

Definition 6. *Let* $\mathcal{L}_{\mathtt{C}}$ *be a propositional language and* $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O}\rangle$ *be a k-valued n-matrix for* $\mathcal{L}_{\mathtt{C}}$. $\mathtt{M}$ *is polarised if, for all assignments* $v\colon \mathtt{Prop} \to \mathtt{Val}$ *and for all formulae* φ *in* $\mathcal{L}_{\mathtt{C}}$, *either for all valuations* $w \supseteq v$ *it holds that* $w(\varphi) \in \mathtt{D}$ *or for all valuations* $w \supseteq v$ *it holds that* $w(\varphi) \notin \mathtt{D}$.

In its most abstract terms, the translation between propositional formulae and modal formulae used in [1] and [2] is defined inductively on the complexity of formulae as follows (recall that in the cited papers the matrix is deterministic).

Definition 7. *Let* $\mathcal{L}_{\mathtt{C}}$ *be a language,* $p \in \mathtt{Prop}$, $\maltese \in \mathcal{L}_{\mathtt{C}}$, *and* $\varphi, \varphi_1, \ldots, \varphi_n$ *be formulae in* $\mathcal{L}_{\mathtt{C}}$. *Let* $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O}\rangle$ *be a k-valued n-matrix with* $\mathtt{Val} = \{\mathbf{1}, \ldots, \mathbf{k}\}$. *Let also* $\bigcirc_1 p, \ldots, \bigcirc_k p$ *be fully modalised formulae whose unique propositional variable is* p. *Then, for each* $i \leq k$, *let*

(1) $\mathbf{i}(p) = \bigcirc_i p$,
(2) $\mathbf{i}(\maltese(\varphi_1, \ldots, \varphi_n)) = \bigvee_{\langle \mathbf{i}_1, \ldots, \mathbf{i}_n\rangle \in \mathbf{i}(f_{\maltese})} [\mathbf{i}_1(\varphi_1) \wedge \ldots \wedge \mathbf{i}_n(\varphi_n)]$, *for* $\mathbf{i}(f_{\maltese}) = \{\langle \mathbf{i}_1, \ldots, \mathbf{i}_n\rangle \in \mathtt{Val}^n \mid \mathbf{i} \in f_{\maltese}(\mathbf{i}_1, \ldots, \mathbf{i}_n)\}$

Moreover, we let $\mathtt{D}(\varphi) = \bigvee_{\mathbf{i} \in \mathtt{D}} (\mathbf{i}(\varphi))$.

For convenience we use the notation $\mathtt{D}(\Gamma)$, for Γ a set of formulae, to indicate that, for each $\psi \in \Gamma$, $\mathtt{D}(\psi)$ holds.

Thanks to the previous definitions, we can state our result.

Theorem 8. *Let* $\models_{\mathbf{Mod}}$ *be a k-good semantics for modal logic. Let* $\mathcal{L}_{\mathtt{C}}$ *be a propositional language and* $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O}\rangle$ *be a polarised k-valued n-matrix for* $\mathcal{L}_{\mathtt{C}}$. *Let* $\Gamma \subseteq \mathcal{L}_{\mathtt{C}}, \varphi \in \mathcal{L}_{\mathtt{C}}$. *Then the following holds:*

$$\Gamma \models_k \varphi \text{ iff } \mathtt{D}(\Gamma) \models_{\mathbf{Mod}} \mathtt{D}(\varphi)$$

A simple example witnesses that the above theorem, under the current definition of translation, does not hold if the n-matrix is not polarised. Though, it remains open whether there exists another translation which allows to prove a similar result for non polarised matrices too.

Moreover, our abstract formulation of the statement above fits also the determinist case, with the unique difference that the result holds for all k-valued matrices, not only the polarised once.

Similarly, all the properties which determine whether a modal semantics is k-good are needed for the theorem to holds under the specific translation. It remains open whether a k-valued non deterministic semantics can be translated into weaker modal semantics. Notice that this question is open also for the deterministic case.

Letting $\bigcirc_1 p = \Diamond p$ and $\bigcirc_2 p = \neg\Diamond p$ if the matrix is two-valued, and $\bigcirc_1 p = \Box p$, $\bigcirc_2 p = \neg\Box p \wedge \Diamond p$ and $\bigcirc_3 p = \neg\Diamond p$ if the matrix is three-valued, one can obtain the following corollary.

Corollary 9. *Let $i \in \{2,3\}$. Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language and $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O} \rangle$ be a polarised i-valued n-matrix for $\mathcal{L}_{\mathtt{C}}$. Let $\Gamma \subseteq \mathcal{L}_{\mathtt{C}}, \varphi \in \mathcal{L}_{\mathtt{C}}$. Then the following holds:*

$$\Gamma \models_i \varphi \Leftrightarrow \mathtt{D}(\Gamma) \models_{\mathbf{S5}} \mathtt{D}(\varphi)$$

Letting $\bigcirc_1 p = \Box p \wedge \Diamond p$, $\bigcirc_2 p = \Box p \wedge \neg\Diamond p$ and $\bigcirc_3 p = \neg\Box p \wedge \Diamond p$, and $\bigcirc_4 p = \neg\Box p \wedge \neg\Diamond p$ if the matrix is four-valued one can obtain the following corollary.

Corollary 10. *Let $\mathcal{L}_{\mathtt{C}}$ be a propositional language and $\mathtt{M} = \langle \mathtt{Val}, \mathtt{D}, \mathtt{O} \rangle$ be a polarised 4-valued n-matrix for $\mathcal{L}_{\mathtt{C}}$. Let $\Gamma \subseteq \mathcal{L}_{\mathtt{C}}, \varphi \in \mathcal{L}_{\mathtt{C}}$. Then the following holds:*

$$\Gamma \models_4 \varphi \Leftrightarrow \mathtt{D}(\Gamma) \models_{\mathbf{N}} \mathtt{D}(\varphi)$$

Finding suitable modal semantics which allow the translation to hold for k-valued semantics, for $k > 4$, is another possible further direction of research.

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References

[1] B. Kooi and A. Tamminga. Three-valued logics in modal logic. *Studia Logica*, 5, 1061–1072, 2013.
[2] E. Kubyshkina. Conservative translations of four-valued logics in modal logic. *Synthese*, 1–17, 2019.

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ON THE QUANTIFIED VERSION OF THE BELNAP–DUNN MODAL LOGIC AND SOME EXTENSIONS OF IT

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The propositional Belnap–Dunn modal logic BK was introduced in [1]; the lattice of its extensions has been studied in [2, 3, 4]. It can be viewed as conservatively expanding the least normal modal logic K by adding 'strong negation' ($\sim$), which allows us to deal with 'gaps' (incomplete information) and 'gluts' (inconsistent information). On the other hand, BK can be obtained by adding material implication ($\to$) and the absurdity constant ($\bot$) to the logic $\mathsf{K}_{\mathsf{FDE}}$ (see [5]). The latter is the least modal expansion of the Belnap–Dunn four-valued logic, also known as *first-degree entailment* and denoted by FDE (see [6, 7]). Intuitively, the semantic values used in FDE are:

(1) `T`, which intuitively stands for 'true';
(2) `F`, which intuitively stands for 'false';
(3) `N`, which intuitively stands for 'neither true nor false';
(4) `B`, which intuitively stands for 'both true and 'false'.

Following [1], we present BK in the language

$$\mathcal{L} := \{\wedge, \vee, \to, \Box, \Diamond, \sim, \bot\}.$$

Below we use the following abbreviations: $\neg\varphi$ stands for $\varphi \to \bot$ and $\varphi \Leftrightarrow \psi$ ('strong equivalence') stands for $(\varphi \leftrightarrow \psi) \wedge (\sim\varphi \leftrightarrow \sim\psi)$. The corresponding deductive system includes the following axiom schemata:

`CL`. all the schemata of propositional classical logic in the language $\{\wedge, \vee, \to, \bot\}$;
`K1`. $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$;
`K2`. $\Box(\varphi \to \varphi)$;
`M1`. $\neg\Box\varphi \leftrightarrow \Diamond\neg\varphi$;
`M2`. $\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$;
`M3`. $\Box\varphi \Leftrightarrow \sim\Diamond\sim\varphi$;
`M4`. $\Diamond\varphi \Leftrightarrow \sim\Box\sim\varphi$;
`SN1`. $\sim\sim\varphi \leftrightarrow \varphi$;
`SN2`. $\sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim\psi)$;
`SN3`. $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$;
`SN4`. $\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$;
`SN5`. $\sim\bot$.

As for the rules, we have *modus ponens* and the monotonicity rules for $\Box$ and $\Diamond$:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\texttt{MP}); \qquad \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}\ (\texttt{MB}); \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}\ (\texttt{MD}).$$

By BK we mean the least set of $\mathcal{L}$-formulas containing the axioms above and closed under the rules `MP`, `MB`, `MD`.

Using the canonical models method, it was proved in [1] that BK and some natural extensions of it are strongly complete w.r.t. appropriate Kripke-style semantics. For instance, a BK-*model* is a triple $\langle \mathcal{W}, v^+, v^- \rangle$ that consists of a standard Kripke frame $\mathcal{W}$ with two independent propositional valuations v^+, v^-. Intuitively, v^+ and v^- correspond to verification and falsification respectively. So we have two relations $\Vdash^+$ and $\Vdash^-$ such that for every propositional variable p:

$$\mathcal{M}, w \Vdash^+ p \ :\Longleftrightarrow\ w \in v^+(p);$$
$$\mathcal{M}, w \Vdash^- p \ :\Longleftrightarrow\ w \in v^-(p).$$

The logical connective $\sim$ is treated as a switch between the processes of verifiability and falsifiability:

$$\mathcal{M}, w \Vdash^+ \sim\varphi \ :\Longleftrightarrow\ \mathcal{M}, w \Vdash^- \varphi;$$
$$\mathcal{M}, w \Vdash^- \sim\varphi \ :\Longleftrightarrow\ \mathcal{M}, w \Vdash^+ \varphi.$$

Since in general no restrictions are imposed on the relationship between verifiability and falsifiability, BK turns out to be paraconsistent. Locally, the truth in BK is four-valued. As well as in FDE, these four values are $\mathtt{T}$, $\mathtt{F}$, $\mathtt{N}$, and $\mathtt{B}$.

It is important that some natural BK-extensions are closely related to Nelson's constructive logics $\mathsf{N3}$ (see [8, 9]) and $\mathsf{N4}^{\perp}$. The latter is a modification proposed by S. P. Odintsov of the logic $\mathsf{N4}$ originally considered in [10]. It was proved in [1] that $\mathsf{N3}$ and $\mathsf{N4}^{\perp}$ can be faithfully embedded into the Belnapian version $\mathsf{BS4}$ of the modal logic $\mathsf{S4}$ and its three-valued extension $\mathsf{B3S4}$ respectively by providing an analogue of the Gödel–McKinsey–Tarski translation of intuitionistic logic into $\mathsf{S4}$.

Although the propositional version of BK is quite well studied, nothing has been previously known about its quantified version. Initially, strong negation appeared in [8] in the first-order setting. It was an approach to an alternative interpretation of negation in intuitionistic arithmetic. This approach was supposed to make intuitionistic negation more constructive, similar to the procedure of falsification instead of the reduction to absurdity. Such arithmetic is now known as Nelson's arithmetic, NA, defined no longer on the basis of intuitionistic first-order logic, but on the basis of Nelson's first-order logic $\mathsf{QN3}$ (see [11]). Since many non-modal logics can be translated into suitable modal logics, the natural question arises: can we do the same for Nelson's logics? Such faithful embeddings can help us in the study of some metamathematical properties of Nelson's logics through the transfer of such properties from the logics in which they are embedded. As we know from the previous paragraph, it can be done in the propositional case. But since $\mathsf{QN3}$ and $\mathsf{QN4}^{\perp}$ are first-order (see [12] for $\mathsf{QN4}^{\perp}$), we would like to find suitable first-order modal logics in which $\mathsf{QN3}$ and $\mathsf{QN4}^{\perp}$ can be embedded.

It would be very natural to expect that the connection between BK-extensions and Nelson's logics should be appropriately inherited to the first-order case. But since the argument of S. P. Odintsov and H. Wansing is based on the completeness of BK-extensions, our first goal is to prove the completeness theorem for the quantified version of BK.

Let σ be a first-order signature. From now on by *σ-formulas* we mean expressions built up from the atomic σ-formulas using the connective symbols of $\mathcal{L}$ and the quantifier symbols $\forall, \exists$ in the usual way. Below we use capital letters for σ-formulas in order to emphasize the difference between the propositional and quantified settings. The deductive system for QBK extends the deductive system for BK by adding the following four axiom schemata:

$\mathtt{Q1}$. $\forall x\, \Phi \to \Phi\,(x/t)$, where t is free for x in Φ;
$\mathtt{Q2}$. $\Phi\,(x/t) \to \exists x\, \Phi$, where t is free for x in Φ;
$\mathtt{Q3}$. $\sim \forall x\, \Phi \leftrightarrow \exists x \sim \Phi$;
$\mathtt{Q4}$. $\sim \exists x\, \Phi \leftrightarrow \forall x \sim \Phi$,

and the Bernays rules:

$$\frac{\Phi \to \Psi}{\Phi \to \forall x\, \Psi}\ (\mathtt{BR1})\,; \qquad \frac{\Phi \to \Psi}{\exists x\, \Phi \to \Psi}\ (\mathtt{BR2})\,.$$

The axioms $\mathtt{Q3}$ and $\mathtt{Q4}$ were originally considered in Nelson's work on constructive arithmetic [8] and were further used in works on Nelson's first-order logics $\mathsf{QN4}^{\perp}$ and $\mathsf{QN3}$. Now denote by Form_σ the set of all σ-formulas and by Sent_σ the set of all σ-sentences (which are formulas with no free variable occurrences, as usual). Given $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \subseteq \mathrm{Form}_\sigma$, we write $\Gamma \vdash_{\mathsf{QBK}} \Delta$ if there is $\Delta' \subseteq \Delta$ such that the disjunction of Δ' can be obtained from the elements of $\Gamma \cup \mathsf{QBK}$ by means of $\mathtt{MP}$, $\mathtt{BR1}$, and $\mathtt{BR2}$.

One of the specific features of both Nelson's logics and QBK is the lack of closure under the usual replacement rule. However, QBK, as well as BK, is closed under the so-called *weak replacement rule*, which is rendered as

$$\frac{\Psi \Leftrightarrow \Phi}{\Theta\,(\Phi/\Psi) \Leftrightarrow \Theta}\ (\mathtt{WR})\,.$$

It is also notable that the negative normal form theorem remains true for QBK, i.e. for any σ-formula Φ there is a σ-formula $\overline{\Phi}$, in which $\sim$ stands only before atomic subformulas, such that $\Phi \Leftrightarrow \overline{\Phi} \in \mathsf{QBK}$ (cf. [13]).

QBK-*models* are first-order Kripke models with expanding domains in which there are two structures with the same domain at each world w: $\mathfrak{A}_w^+$ for verifiability, and $\mathfrak{A}_w^-$ for falsifiability. For any QBK-model $\mathcal{M}$, any world w of this model, and any σ-sentence Φ, the relations $\mathcal{M}, w \Vdash^+ \Phi$ and $\mathcal{M}, w \Vdash^- \Phi$ are defined by induction on the complexity of Φ. If Φ is atomic, we checking its verifiability and falsifiability at the world w by looking at $\mathfrak{A}_w^+$ and $\mathfrak{A}_w^-$ respectively. The case of strong negation is considered as above. Below are the remaining cases:

$$
\begin{array}{rcl}
\mathcal{M}, w \Vdash^{+} \Phi \wedge \Psi & :\Longleftrightarrow & \mathcal{M}, w \Vdash^{+} \Phi \text{ and } \mathcal{M}, w \Vdash^{+} \Psi; \\
\mathcal{M}, w \Vdash^{-} \Phi \wedge \Psi & :\Longleftrightarrow & \mathcal{M}, w \Vdash^{-} \Phi \text{ or } \mathcal{M}, w \Vdash^{-} \Psi; \\
\mathcal{M}, w \Vdash^{+} \Phi \vee \Psi & :\Longleftrightarrow & \mathcal{M}, w \Vdash^{+} \Phi \text{ or } \mathcal{M}, w \Vdash^{+} \Psi; \\
\mathcal{M}, w \Vdash^{-} \Phi \vee \Psi & :\Longleftrightarrow & \mathcal{M}, w \Vdash^{-} \Phi \text{ and } \mathcal{M}, w \Vdash^{-} \Psi; \\
\mathcal{M}, w \Vdash^{+} \Phi \rightarrow \Psi & :\Longleftrightarrow & \mathcal{M}, w \nVdash^{+} \Phi \text{ or } \mathcal{M}, w \Vdash^{+} \Psi; \\
\mathcal{M}, w \Vdash^{-} \Phi \rightarrow \Psi & :\Longleftrightarrow & \mathcal{M}, w \Vdash^{+} \Phi \text{ and } \mathcal{M}, w \Vdash^{-} \Psi; \\
\mathcal{M}, w \Vdash^{+} \Box\Phi & :\Longleftrightarrow & \text{for each } \ u \in W, \text{ if } wRu, \text{ then } \mathcal{M}, w \Vdash^{+} \Phi; \\
\mathcal{M}, w \Vdash^{-} \Box\Phi & :\Longleftrightarrow & \text{there exists } \ u \in W \text{ such that } wRu \text{ and } \mathcal{M}, w \Vdash^{-} \Phi; \\
\mathcal{M}, w \Vdash^{+} \Diamond\Phi & :\Longleftrightarrow & \text{there exists } \ u \in W \text{ such that } wRu \text{ and } \mathcal{M}, w \Vdash^{+} \Phi; \\
\mathcal{M}, w \Vdash^{-} \Diamond\Phi & :\Longleftrightarrow & \text{for each } u \in W, \text{ if } wRu, \text{ then } \mathcal{M}, w \Vdash^{-} \Phi; \\
\mathcal{M}, w \Vdash^{+} \forall x\, \Phi & :\Longleftrightarrow & \text{for each } \ a \in A_w \text{ it holds that } \mathcal{M}, w \Vdash^{+} \Phi\,(x/a)\,; \\
\mathcal{M}, w \Vdash^{-} \forall x\, \Phi & :\Longleftrightarrow & \mathcal{M}, w \Vdash^{-} \Phi\,(x/a) \text{ for some } a \in A_w; \\
\mathcal{M}, w \Vdash^{+} \exists x\, \Phi & :\Longleftrightarrow & \mathcal{M}, w \Vdash^{+} \Phi\,(x/a) \text{ for some } a \in A_w; \\
\mathcal{M}, w \Vdash^{-} \exists x\, \Phi & :\Longleftrightarrow & \text{for each } a \in A_w \text{ it holds that } \mathcal{M}, w \Vdash^{-} \Phi\,(x/a)\,.
\end{array}
$$

The semantic consequence for QBK is defined in the standard way, using the relation $\Vdash^{+}$. We prove that QBK is strongly complete w.r.t. this Kripke-style semantics:

Theorem 1. *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \subseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash_{\mathsf{QBK}} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_{\mathsf{QBK}} \Delta.$$

The proof of strong completeness of QBK is carried out by modifying the canonical models method for first-order modal logics (see [14]). In fact, the resulting modification can be successfully applied to prove the strong completeness of some others quantified modal logics with strong negation. Among the interesting BK-extensions are:

(1) those obtained by excluding either gaps, gluts or both;
(2) those obtained by imposing restrictions (expressible by modal formulas) on accessibility relations in Kripke frames.

Syntactically, the following axiom schemata correspond to the exclusion of gaps and gluts respectively:

`ExM`. $\varphi \vee \sim \varphi$;
`Exp`. $\sim \varphi \rightarrow (\varphi \rightarrow \psi)$.

Intuitively, `ExM` is for excluded middle, and `Exp` is for explosion. As for extensions of the second kind, they are obtained by adding standard axiom schemata expressing suitable properties of accessibility relations. In particular, denote $\mathsf{QBK} + \{\texttt{Exp}\}$ and $\mathsf{QBK} + \{\texttt{ExM}\}$ by $\mathsf{QB3K}$ and QBK° respectively. We prove the following

Theorem 2. $\mathsf{QB3K}$ *and* QBK° *are strongly complete w.r.t. appropriate Kripke-style semantics.*

For the quantified analogs of extensions of the second kind we also obtain similar results. In particular, we prove the strong completeness theorems for the quantified analogs of $\mathsf{BS4}$ and $\mathsf{B3S4}$. Let us denote these by $\mathsf{QBS4}$ and $\mathsf{QB3S4}$ respectively.

In conclusion, we generalize the result that Nelson's logics can be faithfully embedded into appropriate BK-extensions to the first-order setting as follows:

Theorem 3. *The logics* $\mathsf{QN3}$ *and* $\mathsf{QN4}^{\perp}$ *can be faithfully embedded into* $\mathsf{QB3S4}$ *and* $\mathsf{QBS4}$ *respectively.*

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References

[1] S. P. Odintsov, and H. Wansing. Modal logics with Belnapian truth values. *Journal of Applied Non-Classical Logics*, 20(3), 279–301, 2010.

[2] S. P. Odintsov, and E. I. Latkin. BK-lattices. Algebraic semantics for Belnapian modal logics. *Studia Logica*, 100(1-2), 319–338, 2012.

[3] S. P. Odintsov, and S. O. Speranski. The lattice of Belnapian modal logics: special extensions and counterparts. *Logic and Logical Philosophy*, 25(1), 3–33, 2016.
[4] S. P. Odintsov, and S. O. Speranski. Belnap–Dunn modal logics: truth constants vs. truth values. *Review of Symbolic Logic*, 13(2), 416–435, 2020.
[5] G. Priest. Many-valued modal logics: A simple approach. *Review of Symbolic Logic*, 1(2), 190–203, 2008.
[6] N. Belnap. A useful four-valued logic. In Dunn, J. M., & Epstein, G., editors. *Modern Uses of Multiple-Valued Logic, Volume 2 of Episteme,* Reidel, D., 8–37, 1977.
[7] J. M. Dunn. Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, 29(3), 149–168, 1976.
[8] D. Nelson. Constructible falsity. *Journal of Symbolic Logic*, 14(1), 16–26, 1949.
[9] N. N. Vorob'ev. A constructive propositional logic with strong negation (in Russian), *Doklady Akademii Nauk SSSR*, 85(3), 465–468, 1952.
[10] A. Almukdad, and D. Nelson. Constructible falsity and inexact predicates. *Journal of symbolic logic*, 49(1), 231–233, 1984.
[11] Y. Gurevich. Intuitionistic logic with strong negation. *Studia Logica* 36(1), 49–59, 1977.
[12] S. P. Odintsov, and H. Wansing. Inconsistency-tolerant description logic: Motivation and basic systems. In Hendricks, V. F., & Malinowski, J., editors. *Trends in Logic: 50 Years of Studia Logica*. Kluwer Academic Publishers, 301–335, 2003.
[13] S. O. Speranski. Modal bilattice logic and its extensions. *Algebra and Logic*, 60(6), 407-424, 2022.
[14] D. M. Gabbay, V. B. Shehtman, and D. P. Skvortsov. *Quantification in Nonclassical Logic, Volume 1*. Elsevier, 2009.

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ON MODAL FRAGMENTS OF SOME TENSE LOGICS

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The relations between theories formulated in different languages form a subject of numerous research works. In particular simulation results concerning theories in distinct modal languages are also well known (see, eg. [1]). I would like to focus here on some specific issue which arises from studying possible connections between modal and tense operators in formal languages or, more broadly, modal concepts and concepts of time in natural languages.

Historically there are at least two conventional definitions of modal operators in the context of tense[1] logic. One of this, let us denote it as $D1$, postulates $\Box A \equiv GA \wedge A$ and $\Diamond A \equiv FA \vee A$. The second one, $D2$, engages past operators as well: $\Box A \equiv HA \wedge A \wedge GA$ and $\Diamond A \equiv PA \vee A \vee FA$. In [3, 4] the following definition $D3$ is suggested: $\Box A \equiv HGA$ and $\Diamond A \equiv PFA$. Thus a question about modal fragments of tense logics equipped with one of these definitions naturally arises. Formally, a modal logic ML constitutes a modal fragment of a tense logic $TL + D_i$ (where D_i is a definition of modal operators) when $TL + D_i$ is a conservative extension of ML. Some results concerning the modal fragments of the most basic tense logics enriched by definitions $D1$, $D2$ and $D3$ can be found in [3, 4]. For example, modal system **B** (**KTB** in alternative notation) is a modal fragment of tense systems $\mathbf{K_t} + D2$ (likewise $\mathbf{K_t4} + D2$, $\mathbf{K_t4} + \textit{lin-p} + D2$), while **S5** is a modal fragment of $\mathbf{K_t4.3} + D2$ (likewise $\mathbf{K_t4.3} + \textit{noend} + \textit{nobeg} + D2$, $\mathbf{K_t4.3} + \textit{noend} + \textit{nobeg} + \textit{dense} + D2$). On the other hand **B** is a modal fragment of $\mathbf{K_t} + \textit{nobeg} + D3$, while **S5** is a modal fragment of $\mathbf{K_t4} + \textit{lin-p} + \textit{nobeg} + D3$ (and also $\mathbf{K_t4.3} + D3$, $\mathbf{K_t4.3} + \textit{noend} + \textit{nobeg} + D3$, $\mathbf{K_t4.3} + \textit{noend} + \textit{nobeg} + \textit{dense} + D3$)[2].

As is stated in [3], no standard tense logic contains as its modal fragment a modal system which is located between **S4** and **S5**. This observation motivates a certain revision of fundamentals of a minimal tense logic system $\mathbf{K_t}$. From semantic point of view this revision states that past (R_P) and future (R_F) accessibility relations are not considered as inversions of each other as they usually are. Specifically, the following frame conditions are stipulated:

$$R_F(x,y) \Rightarrow R_P(y,x), \tag{1}$$

$$R_F(x,y) \wedge R_P(y,z) \Rightarrow R_F(x,z) \vee R_P(x,z) \vee x = z. \tag{2}$$

The condition (1) corresponds to one of the interconnections axioms of $\mathbf{K_t}$, namely $A \to GPA$. The condition (2) corresponds to schema $FPA \to PA \vee A \vee FA$. Thus the standard interrelation conditions between past and future now changed, so the resulting logic of a class of frames, satisfying (1) and (2) is different from $\mathbf{K_t}$. For this reason it is instructive to distinguish past and future fragments and interconnection conditions of a tense logic. Following [4] let us denote the conditions (1) and (2) by $\underline{S}$ and represent a resulting logic as a pair $\langle KP, KF\rangle_S$, where KP and KF stand for past and future parts of the logic (each can be considered as a notational variant of a modal system **K**). In [3] it is proved that $\langle KP, KF\rangle_S$ is axiomatized by (in addition to the classical tautologies) the following set of schemata (denoted as S): $G(A \to B) \to (GA \to GB)$, $H(A \to B) \to (HA \to HB)$, $A \to GPA$ and $FPA \to PA \vee A \vee FA$. The rules of inference are MP and necessitations for G and H.

By some informal reasons, the system $\langle KP, KF\rangle_S$ was not considered as being interesting with respect to its modal fragments. It was expanded by a schema $A \to PFA$ (which relaxes deterministic flavor of $A \to HFA$) along with $HA \to PA$ and a corresponding semantic condition

$$\exists y (R_P(x,y) \wedge R_F(y,x)). \tag{3}$$

The resulting system is denoted as $\langle DP, KF\rangle_R$, where $R = S + A \to PFA$, $DP = KP + HA \to PA$, $\underline{R} = (1) + (2) + (3)$. This is the basis for constructing a family of tense logics enriched by $D3$ and containing some commonly known modal systems like **S4.2**, **S4.3**.

---

[1] I prefer to use an old fashioned term "tense logic" rather then "temporal logic" since the languages of the systems considered here based on Prior-style tense modalities.

[2] To save space I use the notations for well known axiom schemata as they given in [2].

Nevertheless, it seems natural to consider even more unusual tense logic of a class of frames, specified by the following pair of conditions:

$$R_P(x,y) \wedge R_F(y,z) \Rightarrow R_P(x,z) \vee R_P(x,z) \vee x = z, \tag{4}$$

$$R_F(x,y) \wedge R_P(y,z) \Rightarrow R_F(x,z) \vee R_P(x,z) \vee x = z. \tag{5}$$

Let us denote (4)+(5) by $\underline{Q}$. The corresponding pair of axiom schemata, $FPA \to PA \vee A \vee FA$ and $PFA \to PA \vee A \vee FA$, is denoted by Q. It can be verified that logic $\langle KP, KF\rangle_Q$ is determined by a class of frames satisfying the conditions (4) and (5).

Now we can determine a modal fragment of the system $\langle KP, KF\rangle_Q + Di$ ($i \in 1,2,3$) itself and study the modal fragments of its successive modifications. It is clear that there are lots of possible extensions of the basic system $\langle KP, KF\rangle_Q$ with respect to its "internal" parts KP and KF and interconnection conditions. On the one hand we can incorporate into "internal" parts the common systems of tense logic representing different properties of flow of time, on the other hand interconnection conditions can help to capture a particular modal system as a modal fragment. Here we inspect just some examples leaving the larger picture for a lengthy paper.

Let us say a few words concerning the method of proving statements about modal fragments in general. There are at least three ways of addressing this problem. One of these methods is purely semantical (and is often used in [3]), the second one is based on embedding techniques. Finally one can compare theories using definitional approach (which in some sense is related to embeddings but do not use recursive definitions for translating the whole language). The brief idea of the latter is as follows. Suppose that there are two theories, ML and TL, formulated in different languages. Let D_{TL} denote the definition in ML the terms of TL absent from ML. Likewise D_{ML} is a definition of terms from ML within TL. Now it can be shown that $TL + D_{ML}$ is a conservative extension of ML if there is a proof of two assertions: (α) ML is a subsystems of $TL + D_{ML}$ and (β) $TL + D_{ML}$ is a subsystem of $ML + D_{TL}$. Is is convenient to use axiomatic representations of the systems to prove (α) and (β). It is also important to note that one must prove definitions D_{ML} within $ML + D_{TL}$ to accomplish the proof for the proposition (β). Moreover, if some system $TL_1 + D_{ML}$ is an extension of $TL + D_{ML}$ but contained in $ML + D_{TL}$, then $TL_1 + D_{ML}$ is a conservative extension of ML. Next we formulate some results concerning the basic system and some of its extensions.

Theorem 1. *The following statements hold:*

(1) **K** *is a modal fragment of any tense system* TL *such that*

$$\langle KP, KF\rangle_Q + D3 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{K} + GA \equiv A + HA \equiv \Box A.$$

(2) **S4** *is a modal fragment of any tense system* TL *such that*

$$\langle KP, KF4\rangle_Q + D1 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{S4} + GA \equiv \Box A + HA \equiv \Box A.$$

(3) **S4** *is a modal fragment of any tense system* TL *such that*

$$\langle KP4, KF4\rangle_Q + D2 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{S4} + GA \equiv \Box A + HA \equiv \Box A.$$

(4) **K4** *is a modal fragment of any tense system* TL *such that*

$$\langle KP, KF4\rangle_Q + D3 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{K4} + GA \equiv A + HA \equiv \Box A.$$

Note that some modifications of KP and KF parts are necessary, while interconnection conditions are constant. But in more complex cases we have to modify the latter conditions as well. Let us consider the case of **K4.3** system. We construct a tense logic equipped by $D3$ which contains **K4.3** as a modal fragment.

Recall that $DP4$ means $KP + HA \to PA + HA \to HHA$. We also use the standard tense logic versions of the axiom **3**, that is $DP4.3 = DP4 + H(A \wedge HA \to B) \vee H(B \wedge HB \to A)$, $KF4.3 = KF + GA \to GGA + G(A \wedge GA \to B) \vee G(B \wedge GB \to A)$. Next we replace Q with the following three schemata:

$$F(B \wedge PA) \rightarrow PA \vee A \vee F(A \wedge FB), \tag{6}$$

$$P(B \wedge FA) \rightarrow FA \vee A \vee P(A \wedge PB), \tag{7}$$

$$FA \wedge PB \rightarrow P(B \wedge FA). \tag{8}$$

Let $Q3$ to denote this modified list of expressions. The corresponding first order frame conditions can be readily seen:

$$R_F(x,y) \wedge R_P(y,z) \rightarrow (R_P(x,z) \vee x = z \vee (R_F(x,z) \wedge R_F(z,y))), \tag{9}$$

$$R_P(x,y) \wedge R_F(y,z) \rightarrow (R_F(x,z) \vee x = z \vee (R_P(x,z) \wedge R_P(z,y))), \tag{10}$$

$$R_P(x,y) \wedge R_F(x,z) \rightarrow R_F(y,z). \tag{11}$$

Let us put $\underline{Q3} = (9) + (10) + (11)$ in addition to $\forall x \exists y R_P(x,y)$, transitivity for R_P and R_F along with conditions for past and future linearity with respect to the corresponding relations, that is $R_P(x,y) \wedge R_P(x,z) \rightarrow R_P(y,z) \vee y = z \vee R_P(z,y)$ and its mirror image for F. It can be verified that a tense logic $\langle DP4.3, KF4.3\rangle_{Q3}$ is determined by the class of frames characterized by conditions $\underline{Q3}$. Now we have

Theorem 2. **K4.3** *is a modal fragment of any tense system TL such that*

$$\langle DP4.3, KF4.3\rangle_{Q3} + D3 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{K4.3} + GA \equiv A + HA \equiv \Box A.$$

As one can see, the study of modal fragments of tense logics may follow two general strategies. On the one hand one can try to capture a modal logic by some, more or less artificially constructed, tense logic. More difficult and challenging problem is to find the modal fragments of some standard tense systems such as linear time logics of naturals, integers, rationals or reals extended with some definition of modal operators.

References

[1] M. Kracht, F. Wolter. Simulation and Transfer Results in Modal Logic – A Survey. Studia Logica, 59, 149–177, 1997.

[2] V. Goranko, A. Rumberg, Temporal Logic, *The Stanford Encyclopedia of Philosophy (Summer 2022 Edition)*, Edward N. Zalta (ed.), URL=https://plato.stanford.edu/archives/sum2022/entries/logic-temporal .

[3] V. A. Smirnov. The definition of modal operators by means of tense operators. *Intensional Logic: Theory and Applications. Acta Philosophica Fennica*, Vol. 35, 50–69, 1982. (the same paper in Russian in *Modal and intensional logics and their application to the problems of methodology of science*, Moscow, 1984.)

[4] V. A. Smirnov. Tense logics with nonstandard interconnections between past and future. *Modal and intensional logics. Proceedings of VIII conference "Logic and methodology of science", Vilnius, 1982.* In Russian.

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EXPERIMENTAL PROVER FOR TOPE LOGIC

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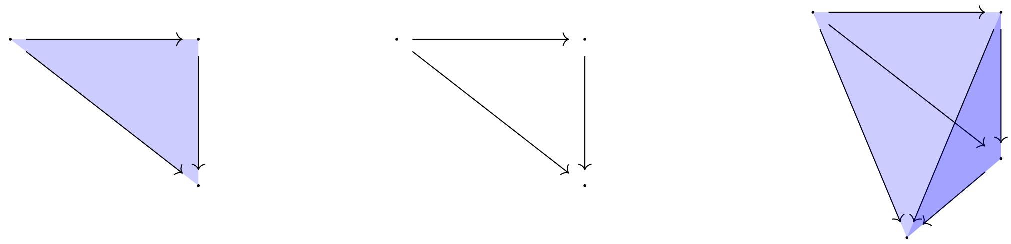
Tope logic comprises the top two layers of Riehl and Shulman's type theory for synthetic ∞-categories [3] (we will refer to this type theory as RSTT[1]). Reasoning in higher categories involves (higher-dimensional) diagrams, and, in RSTT, cubes and topes provide a synthetic approach to making shapes of diagrams and reason about relationships between such shapes, most importantly, involving inclusion of shapes.

The tope logic consists of two layers — cube layer and tope[2] layer. The cube layer is an intuitionistic type theory with products, providing abstract spaces to work in. For example, cube $\mathbb{2}$ is a directed interval with two distinct points $\mathbf{0} : \mathbb{2}$ and $\mathbf{1} : \mathbb{2}$. The interval is abstract in the sense that a point $t : \mathbb{2}$ is not necessarily one of $\mathbf{0}$ and $\mathbf{1}$. The directedness of the interval $\mathbb{2}$ is specified in the tope layer:

$$
\begin{aligned}
x : \mathbb{2} \mid \cdot &\vdash (x \leq x) \\
x : \mathbb{2}, y : \mathbb{2}, z : \mathbb{2} \mid (x \leq y), (y \leq z) &\vdash (x \leq z) \\
x : \mathbb{2}, y : \mathbb{2} \mid (x \leq y), (y \leq x) &\vdash (x \equiv z) \\
x : \mathbb{2}, y : \mathbb{2} \mid \cdot &\vdash (x \leq y) \vee (y \leq x) \\
x : \mathbb{2} \mid \cdot &\vdash (0 \leq x) \\
x : \mathbb{2} \mid \cdot &\vdash (x \leq 1) \\
\cdot \mid (0 \equiv 1) &\vdash \bot
\end{aligned}
$$

Proof search for full intuitionistic logic with equalities ($\mathbf{LJ}^{\simeq}$) has been studied by Voronkov [5]. However, tope logic is a two-layer formal system and we cannot apply tools for $\mathbf{LJ}^{\simeq}$ immediately. Additionally, tope logic appears to have a simpler core so might admit a more straightforward algorithm.

The tope layer allows then to define restrictions on such abstract spaces, effectively carving out shapes inside cubes. Topes are given as formulas in intuitionistic logic over the cube layer (i.e. under a context with point variables). Some common examples of 2- and 3-dimensional shapes are presented in Fig. 1.



(A) 2-dim simplex, notated Δ^2: $\{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid s \leq t\}$

(B) Boundary of Δ^2, notated $\partial\Delta^2$: $\{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid s \equiv \mathbf{0} \vee s \equiv t \vee t \equiv \mathbf{1}\}$

(C) A 3-dim horn, notated Λ^3_2: $\{\langle t_1, t_2, t_3\rangle : \mathbb{2}^3 \mid (t_3 \equiv \mathbf{0} \wedge t_2 \leq t_1) \vee (t_3 \leq t_2 \wedge t_2 \equiv t_1) \vee (t_3 \leq t_2 \wedge t_1 \equiv \mathbf{1})\}$

FIGURE 1. Examples of shapes.

Starting with different basic cubes and topes, we can construct different kinds of diagrams. In particular, Riehl and Shulman [3, Section 3] develop a simplicial type theory for synthetic ∞-categories starting from a directed interval tope, but also mention a possible formulation for a cubical type theory. The cubical interval is used together with the simplicial interval in the work of Streicher and Weinberger [4].

[1] Riehl and Shulman's Type Theory

[2] Riehl and Shulman use the term "tope", thinking of topes as "*polytopes* embedded in cubes", and we inherit their terminology

Taking the first two layers of RSTT and varying basic cubes and topes we obtain a family of tope logics. The cube layer is a simple intuitionistic logic with products, possibly extended with user-defined point constructors. The tope layer is an intuitionistic logic over the cube layer, possibly extended with user-defined axioms. Importantly, tope layer includes an equality tope, which is seen from the type layer in RSTT as definitional equality.

Well-formedness of terms in RSTT depends upon proofs in the tope layer. Such proofs are often left implicit on paper and are considered trivial. However, when automating reasoning in RSTT, these checks have to be performed by a computer, which motivates design of a formal system to prove theorems in tope logics.

In this paper, we report on a work-in-progress for developing an experimental prover for tope logic (in general, and for special cases of simplicial and cubical logic), extensible with user-defined cubes and topes, and discuss preliminary results.

Experimental Theorem Prover

We have implemented an experimental prover for the tope logic, with user-defined cubes and topes. We used Haskell programming language with heavy use of `logict` library [2], for backtracking and interleaving computations used for the proof search. The source code of the implementation is available on GitHub at `https://github.com/fizruk/simple-topes` and an interactive online version of the prover is available at `https://fizruk.github.io/simple-topes/`.

The theorem prover supports tope logic as defined in RSTT [3, Figures 1 and 2] as well as user-defined cubes and topes. For example, the definitions for simplicial logic [3, Section 3] are presented in Fig. 2.

The prover currently implements a version of reasoning using sequent calculus for tope logic with a cut rule, to admit arbitrary user-defined inference rules, as well as (a limited form of) equality saturation used for the equality tope ($\equiv$). We note the following properties of the experimental implementation:

(1) User-defined axioms are formulated in the form of sequent calculus inference rules.
(2) The order of the rules affects the proof search.
(3) All user-defined rules as considered by the cut rule, except
   (a) rules where right hand side is a tope (propositional) variable or
   (b) rules where right hand side is already a trivial consequence of current premises in $\mathbf{LJ}^{\simeq}$.

The last heuristic speeds up proof search significantly, allowing to avoid unnecessary branching when applying rules such as excluded middle for the inequality tope.

A simplified version of reasoning in tope logic has also been integrated in an experimental proof assistant for synthetic ∞-categories, RZK (available at `https://github.com/fizruk/rzk`). A work-in-progress formalisation of Riehl and Shulman's original paper [3] is available at `https://github.com/fizruk/sHoTT`. The current implementation of tope logic in RKZ has proven sufficient for these formalisations. In the future, the project aims to encompass formalisations of synthethic fibered ∞-categories from the work of Buchholtz and Weinberger [1] as well as explore the non-simplicial part of the type theory with shapes, such as cubical type theory [3, Remark 3.2]. For these, RZK is planned to be extended with support for user-defined cubes and topes.

An interesting feature of RZK proof assistant is the ability to render diagrams that correspond to certain proof terms and theorem. In Fig. 3, the proof term is constructed by taking a subshape corresponding to an internal face of a prism. In the formalisation, this is given by an mapping 2D coordinates `(t, s)` into 3D coordinates `((t, t), s)`. The diagram on the right highlights the embedding in red. To implement such highlighting, it is necessary to understand, which parts of the 3D space correspond to the image of this mapping. In general, given a mapping $f : I \to J$ and a constraints $t : I \vdash \psi\ \mathtt{tope}$ and $s : J \vdash \phi\ \mathtt{tope}$, we want to solve $s : J \mid \cdot \vdash \exists (t : I).\ \psi \wedge \phi$. Thus, there is a need to extend the logic and its solvers with support for existential quantification. Currently, the implementation in RZK relies on a saturation-based heuristic to solve such cases of existential quantification. However, a proper support for existential quantification is required in future.

The experimental provers currently lack a proper formalisation for the proof search algorithm with any soundness and completeness results. The implementation also does not include type checking at the moment, which would help with directing and optimising the proof search. Finally, representation of propositions is not efficient and can be changed to improve overall performance of the prover.

```
-- | The strict interval cube (see RS17 Section 3.1).
cube 2 with
  point 0 -- ^ 0 point (left point).
  point 1 -- ^ 1 point (right point).

-- | Inequality tope for the strict interval cube.
tope ≤(2, 2) with
  rule "(≤) distinct" where
    ─────────────
    · | 0 ≡ 1 ⊢ ⊥

  rule "(≤) antisymmetry" where
    ───────────────────────────────────────
    t : 2, s : 2 | ≤(t, s), ≤(s, t) ⊢ s ≡ t

  rule "(≤) transitivity" where
    ────────────────────────────────────────────────
    t : 2, s : 2, u : 2 | ≤(t, s), ≤(s, u) ⊢ ≤(t, u)

  rule "(≤) excluded middle" where
    ───────────────────────────────────────
    t : 2, s : 2 | · ⊢ ≤(t, s) ∨ ≤(s, t)

  rule "(≤) one" where
    ───────────────────
    t : 2 | · ⊢ ≤(t, 1)

  rule "(≤) zero" where
    ───────────────────
    t : 2 | · ⊢ ≤(0, t)

  rule "(≤) reflexivity" where
    ───────────────────────
    t : 2 | · ⊢ ≤(t, t)
```

FIGURE 2. Definition of a directed interval cube $\mathbb{2}$, inequality tope $(\leq)$, and corresponding axioms in sequent calculus form.

```
#def face
  (A : U)
  (x y z : A)
  (f : hom A x y)
  (a : Sigma (g : hom A y z),
    {((t1, t2), t3) : 2 * 2 * 2 | t3 <= t1 \/ t2 <= t1}
    -> A [ t1 === 0_2 |-> f t2, t1 === 1_2 |-> g t3 ])
  : Δ² -> A
  := \(t, s) -> second a ((t, t), s)
```

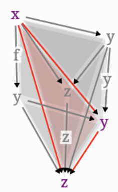


FIGURE 3. A definition in RZK and a corresponding automatically rendered 3D diagram.

REFERENCES

[1] Ulrik Buchholtz and Jonathan Weinberger. *Synthetic fibered $(\infty, 1)$-category theory.* To appear in *Higher Structures*. 2022. arXiv: 2105.01724 [math.CT].

[2] Oleg Kiselyov et al. "Backtracking, Interleaving, and Terminating Monad Transformers: (Functional Pearl)". In: *SIGPLAN Not.* 40.9 (Sept. 2005), pp. 192–203. ISSN: 0362-1340. DOI: 10.1145/1090189.1086390. URL: https://doi.org/10.1145/1090189.1086390.

[3] Emily Riehl and Michael Shulman. "A type theory for synthetic ∞-categories". In: *Higher Structures* 1 (1 2017). arXiv: 1705.07442 [math.CT].

[4] Thomas Streicher and Jonathan Weinberger. *Simplicial sets inside cubical sets.* 2021. arXiv: 1911.09594 [math.CT].

[5] Andrei Voronkov. "Proof-Search in Intuitionistic Logic with Equality, or Back to Simultaneous Rigid E-Unification". In: *J. Autom. Reason.* 30.2 (2003), pp. 121–151. DOI: 10.1023/A:1023260415982. URL: https://doi.org/10.1023/A:1023260415982

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MANY-DIMENSIONAL MODAL LOGICS FOR NEIGHBORHOOD SEMANTICS

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In 2003 a very influential book called "Many-Dimensional Modal Logics: Theory and Applications" was published [2]. This book mainly considered modal logics in the context of Kripke semantics. But in mathematics, computer science and philosophy not all structures can be presented as Kripke frames. In resent year more and more researchers turn their attention to topological and neighborhood semantics. These semantics are more suited for modeling all sorts of continuous processes and structures. Unfortunately, many-dimentional modal logics in the context of topological and neighborhood semantics is still very underresearched.

A *neighborhood frame* (or an n-frame) is a pair $\mathcal{X} = (X, \tau)$, where X is a nonempty set and $\tau : X \to 2^{2^X}$ is the *neighborhood function* of $\mathcal{X}$. Sets from $\tau(x)$ are called *neighborhoods* of point x. *A neighborhood model* (n-model) is a pair $M = (\mathcal{X}, V)$, where $\mathcal{X} = (X, \tau)$ and V is a valuation on X, i.e. a function from the set of all propositional variables to set 2^X. The truth relation $\models$ on n-models for logical connectives is defined in the classical way and for modalities the definition is the following:

$$M, x \models \Box A \iff \exists U \forall y (y \in U \in \tau(x) \Rightarrow M, y \models A).$$

A formula A is *valid* on an n-frame $\mathcal{X}$ if it is true at all points on all models based on $\mathcal{X}$ (Notation: $\mathcal{X} \models A$). For a class of n-frames $\mathcal{C}$ formula A is *valid* on $\mathcal{C}$ if $\forall \mathcal{X} \in \mathcal{C} (\mathcal{X} \models A)$. The set of all formulas that are valid on a given frame (a class of frames) is called the logic of this frame (class of frames). Notations: $Log(F)$ and $Log(\mathcal{C})$ respectively.

Any logic of a Kripke frame is normal, i.e. it contains axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and is closed under the necessitation rule $(\frac{A}{\Box A})$. Whereas logic of an n-frame is not necessarily normal. For it to be normal $\tau(x)$ has to be a filter for any $x \in \mathcal{X}$. *Filter* is a nonempty set of sets closed under intersections and supersets. From now on we will assume that all n-frames are like this.

N-frames is a more general notion then a Kripke frame. Given a Kripke frame $F = (W, R)$ we can construct an n-frame $\mathcal{N}(F) = (W, \tau_R)$, such that $\tau_R(x) = \{U \subseteq W \mid R(x) \subseteq U\}$. Then the validity of formulas will be preserved by this operation.

One of the natural construction that "increase" dimension is the product. For two Kripke frames $F_1 = (W_1, R_1)$ and $F_2 = W_2, R_2$ their product is a Kripke frame with 2 relations $F_1 \times F_2 = (W_1 \times W_2, R_1^h, R_2^v)$, where

$$\begin{aligned} (x, y) R_1^h (z, t) &\iff x R_1 z \;\&\; y = t; \\ (x, y) R_2^v (z, t) &\iff x = z \;\&\; y R_2 t. \end{aligned}$$

In a similar way we can define the product of two n-frames. Let $\mathcal{X}_1 = (X_1, \tau_1)$ and $\mathcal{X}_2 = (X_2, \tau_2)$ be two n-frames. Their product is an n-frame with 2 neighborhood functions:

$$\begin{aligned} &\mathcal{X}_1 \times \mathcal{X}_2 = (X_1 \times X_2, \tau_1^h, \tau_2^v), \text{ where} \\ &\tau_1^h(x_1, x_2) = \{U \subseteq X_1 \times X_2 \mid \exists V (V \in \tau_1(x_1) \;\&\; V \times \{x_2\} \subseteq U)\}; \\ &\tau_2^v(x_1, x_2) = \{U \subseteq X_1 \times X_2 \mid \exists V (V \in \tau_2(x_2) \;\&\; \{x_1\} \times V \subseteq U)\}. \end{aligned}$$

In frames with several relations (neighborhood functions) we use modalities with corresponding sub indexes.

For two unimodal logics L_1 and L_2, we define the *n-product* of them as a logic with two modalities:

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = Log\big(\{\mathcal{X}_1 \times \mathcal{X}_2 \mid \mathcal{X}_1 \models \mathsf{L}_1 \;\&\; \mathcal{X}_2 \models \mathsf{L}_2\}\big).$$

The fusion of L_1 and L_2 is the minimal 2-modal logic $\mathsf{L}_1 * \mathsf{L}_2$ axiomatized with axioms of L_1 rewritten with $\Box_1$ and axioms of L_2 rewritten with $\Box_2$.

Let $\mathcal{KF} = \{\mathcal{N}(F) \mid F - \text{Kripke frame}\}$. The product of modal logics defined in [2] and earlier in [10] can be defined as an n-product with restrictions on the class of n-frames:

$$\mathsf{L}_1 \times \mathsf{L}_2 = Log\big(\{\mathcal{X}_1 \times \mathcal{X}_2 \mid \mathcal{X}_1 \models \mathsf{L}_1, \mathcal{X}_2 \models \mathsf{L}_2 \;\&\; \mathcal{X}_1, \mathcal{X}_2 \in \mathcal{KF}\}\big).$$

If we restrict only one component of the product we get a mixed neighborhood-Kripke product (cf. [6]):

$$\mathsf{L}_1 \times_{nk} \mathsf{L}_2 = Log(\{\mathcal{X}_1 \times \mathcal{X}_2 \mid \mathcal{X}_1 \models \mathsf{L}_1, \mathcal{X}_2 \models \mathsf{L}_2 \ \&\ \mathcal{X}_2 \in \mathcal{KF}\}).$$

This observation shows that

$$\mathsf{L}_1 * \mathsf{L}_2 \subseteq \mathsf{L}_1 \times_n \mathsf{L}_2 \subseteq \mathsf{L}_1 \times_{nk} \mathsf{L}_2 \subseteq \mathsf{L}_1 \times \mathsf{L}_2.$$

It worth noting that these definitions are correct for monotone logics. Product of n-frames first appeared in [9], where Sano studied the product of hybrid monotone modal logics, i.e. modal logics with nominals and satisfaction operators $@_i$. Nobody studied products of pure monotone modal logics jet.

Topological semantics is a particular case of neighborhood semantics for extensions of $\mathsf{S4}$. For a topological space $\mathfrak{X} = (X, T)$ we can define neighborhood function

$$\tau(x) = \{U \mid \exists U' \in T(x \in U' \subseteq U)\}.$$

The truth relation on $\mathfrak{X}$ and on $\mathcal{X} = (X, \tau)$ will be the same. First paper where topological product was defined was [11].

The *derivational* topological semantics (d-semantics) is also a particular case of neighborhood semantics. For a topological space $\mathfrak{X} = (X, T)$ we can define neighborhood function

$$\tau_d(x) = \{U \mid \exists U' \in T(x \in U' \ \&\ U' \setminus \{x\} \subseteq U)\}.$$

The truth relation on $\mathfrak{X}$ with d-semantics and on $\mathcal{X} = (X, \tau_d)$ will also be the same. The results on the d-products of modal logics (the d-logic of the products of topological spaces) can be found in [5].

Let us put all known results from [1, 4, 6, 7, 11] into a table:

L_1	L_2	$\mathsf{L}_1 \times_n \mathsf{L}_2$	$\mathsf{L}_1 \times_{nk} \mathsf{L}_2$	$\mathsf{L}_1 \times \mathsf{L}_2$
$\mathsf{S4}$	$\mathsf{S4}$	$\mathsf{S4} * \mathsf{S4}$	$\mathsf{S4} * \mathsf{S4} + chr + com_{\to}$	$\mathsf{S4} * \mathsf{S4} + chr + com_{\leftrightarrow}$
$\mathsf{S4}$	$\mathsf{S5}$	$\mathsf{S4} * \mathsf{S5} + chr + com_{\to}$	$\mathsf{S4} * \mathsf{S5} + chr + com_{\to}$	$\mathsf{S4} * \mathsf{S5} + chr + com_{\leftrightarrow}$
$\mathsf{S5}$	$\mathsf{S5}$	$\mathsf{S5} * \mathsf{S5} + chr + com_{\leftrightarrow}$	$\mathsf{S5} * \mathsf{S5} + chr + com_{\leftrightarrow}$	$\mathsf{S5} * \mathsf{S5} + chr + com_{\leftrightarrow}$
HTC-logic $+\Diamond\top$	HTC logic $+\Diamond\top$	$\mathsf{L}_1 * \mathsf{L}_2$	$\mathsf{L}_1 * \mathsf{L}_2 + chr + com_{\to}$	$\mathsf{L}_1 * \mathsf{L}_2 + chr + com_{\leftrightarrow}$
HTC logic	HTC-logic	$\mathsf{L}_1 * \mathsf{L}_2 + \Delta$	$\mathsf{L}_1 * \mathsf{L}_2 + chr + com_{\to} + \Delta_1$	$\mathsf{L}_1 * \mathsf{L}_2 + chr + com_{\leftrightarrow}$
HTC logic $+\Diamond\top$	HTC-logic	$\mathsf{L}_1 * \mathsf{L}_2 + \Delta_2$	$\mathsf{L}_1 * \mathsf{L}_2 + chr + com_{\to}$	$\mathsf{L}_1 * \mathsf{L}_2 + chr + com_{\leftrightarrow}$
$\mathsf{S4.1}$	$\mathsf{S4}$	$\mathsf{S4.1} * \mathsf{S4} + A_{mk2}$	?	?

Notation:

$$\Delta_1 = \{\phi \to \Box_1\phi \mid \phi \text{ is closed and } \Box_1\text{-free}\}, \qquad com_{\to} = \Box_1\Box_2 p \to \Box_2\Box_1 p,$$

$$\Delta_2 = \{\psi \to \Box_2\psi \mid \psi \text{ is closed and } \Box_2\text{-free}\}, \qquad com_{\leftrightarrow} = \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p,$$

$$\Delta = \Delta_1 \cup \Delta_2, \qquad chr = \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p,$$

$$\mathsf{S4.1} = \mathsf{S4} + \Diamond(\Diamond p \to \Box p) \qquad A_{mk2} = \Diamond_1\Box_2(\Diamond_1 p \to \Box_1 p).$$

A logic L is called *HTC-logic* (Horn preTransitive Closed logic) if it can be axiomatized by closed formulas and formulas of the type $\Box p \to \Box^n p$ or $p \to \Box^n p$.

Some results for nk-products have not published jet.

Other lines of research related to the many-dimensional modal logics is the quantified modal logics and the intuitionistic modal logics. This happens because the two-dimensional structures are well suited for semantics for these logics.

In [8] it was proved that for any HTC-logic L its quantified counterpart QL is complete w.r.t. n-frames with constant domain. This is quite surprising since QL is not complete w.r.t. Kripke frames with constant domain and only w.r.t. to Kripke frames with expanding domains.

For intuitionistic modal logic, our work is in progress but we expect similar completeness results.

Another important aspect of this research is the complexity of the logics. In [3] it was proved that the Kripke frame based products of transitive modal logics in many cases (e.g. $\mathsf{S4}, \mathsf{K4}, \mathsf{S4.1}$ etc.) are undecidable. On the other hand, the fusion of decidable logics is decidable (see [2]). Adding Δ preserves the decidability of logic. So the n-products of logics $\mathsf{S4}, \mathsf{K4}, \mathsf{D4}$ are decidable.

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References

[1] D. Gabbay and V. Shehtman. Products of modal logics. Part I. *Journal of the IGPL*, 6:73–146, 1998.

[2] D.M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics : theory and applications*, volume 148 of *Studies in logic and the foundations of mathematics*. Elsevier, 2003.

[3] D. Gabelaia, A. Kurucz, F. Wolter, and M. Zakharyaschev. Products of 'transitive' modal logics. *The Journal of Symbolic Logic*, 70(3):993–1021, 2005.

[4] P. Kremer. The topological product of S4 and S5. *Unpublished*, 2011.
[5] A. Kudinov. D-logic of product of rational numbers. In *Procedings of Information Technology and Systems*, pages 95–99, Moscow, 2013.
[6] A. Kudinov. Neighborhood-kripke product of modal logics. *Topology, Algebra, and Categories in Logic (TACL'17)*, 2017.
[7] A. Kudinov. On neighbourhood product of some horn axiomatizable logics. *Logic Journal of the IGPL*, 26(3):316–338, 2018.
[8] A. Kudinov. Neighbourhood completeness for quantified pretransitive modal logics. arXiv, 2021.
[9] K. Sano. Axiomatizing hybrid products of monotone neighborhood frames. *Electr. Notes Theor. Comput. Sci.*, 273:51–67, 2011.
[10] V. Shehtman. Two-dimensional modal logic. *Mathematical Notices of USSR Academy of Science*, 23:417–424, 1978. (Translated from Russian).
[11] J. van Benthem, G. Bezhanishvili, B. Cate, and D. Sarenac. Multimodal logics of products of topologies. *Studia Logica*, 84:369–392, 2006.

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SOME ALGEBRAIC SEMANTICS FOR SUPERINTUITIONISTIC FIRST-ORDER LOGICS

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1. Short Abstract

There is a variety of different semantics for intuitionistic and superinuitionistic predicate logics. The Algebraic Kripke sheaf semantics was introduced in [2] by N.-Y. Suzuki. The semantics of Heyting-valued structures was introduced in [3] by A. Dragalin and was further studied in [1] by D. Gabbay, V. Shehtman and D. Skvortsov. This work shows that Suzuki's algebraic Kripke sheaf semantics is reducible to Heyting-valued structure semantics. This means that a sheaf of algebras in Suzuki's model can be "substituted" by a single algebra which is associated with the root of that model.

2. Algebraic Kripke Sheaf Semantics

This semantics was defined by N.-Y. Suzuki in [2]. Consider a (super-)intuitionistic predicate logic without equality with universal and existential quantifiers. Then we define an **Algebraic Kripke Sheaf** model as follows:

Definition 1. *Let $\boldsymbol{M} = \langle M, \preccurlyeq \rangle$ be a partially ordered set with the least element $0^M \in \boldsymbol{M}$ and D be a covariant functor from $\boldsymbol{M}$ to the category of all sets denoted by SET, which means that:*

- *for every $a \in M$ the set $D(a)$ is non-empty;*
- *for every $a, b \in M$ with $a \preccurlyeq b$ there exists a mapping $D_{ab} : D(a) \to D(b)$;*
- *$D_{aa} = id_{D(a)}$ for every $a \in M$;*
- *$D_{ab} \circ D_{bc} = D_{ac}$ for every $a, b, c \in M$ with $a \preccurlyeq b \preccurlyeq c$.*

The functor D is called a domain-sheaf over $\boldsymbol{M}$, the triple $\langle M, \preccurlyeq, 0^M \rangle$ is called a Kripke base and the pair $\langle \boldsymbol{M}, D \rangle$ is called a Kripke sheaf.

For each $a, b \in M$ and $d \in D(a)$, element $D_{ab}(d)$ is said to be an inheritor of an element d at the point b. For each sentence A assigned with elements from $D(a)$ and each element $b \in M$ such that $b \succcurlyeq a$ we define an inheriting sentence A_{ab} obtained by replacing every occurrence of every $u \in D(a)$ by its inheritor $v = D_{ab}(u)$ at (b).

Definition 2. *Let $\mathcal{H}$ denote a category of non-degenerate complete Heyting algebras and complete monomorphisms. Consider a Kripke sheaf $\langle \langle M, \preccurlyeq, 0^M \rangle, D \rangle$. A contravariant functor P from a Kripke base $\boldsymbol{M} = \langle M, \preccurlyeq, 0^M \rangle$ to $\mathcal{H}$ is called a Heyting-valued-sheaf over M and $\mathcal{K} = \langle \boldsymbol{M}, D, P \rangle$ is called an algebraic Kripke sheaf.*

Definition 3. *Let $AF_{D(a)}$ be a set of all atomic sentences in the language extended with elements of the set $D(a)$ for some $a \in M$. A mapping V which assigns each pair (a, A) where $a \in M$ and $A \in AF_{D(a)}$ to an element of $P(a)$ is said to be a valuation on $\langle \boldsymbol{M}, D, P \rangle$ if for every $a, b \in \boldsymbol{M}$ with $a \preccurlyeq b$ we have $V(a, A) \leq P_{ab}\left(V\left(b, A_{ab}\right)\right)$*

Definition 4. *We extend V to a mapping which assigns an element of non-degenerate complete Heyting-Algebra algebra $P(a) = \langle P(a), \vee^{P(a)}, \wedge^{P(a)}, \to^{P(a)}, 0^{P(a)}, 1^{P(a)} \rangle$ to each pair of an element $a \in M$ and a sentece A of the language extended with constants from $D(a)$ inductively as follows:*

- $V(a, A \wedge B) = V(a, A) \wedge^{P(a)} V(a, B)$;
- $V(a, A \vee B) = V(a, A) \vee^{P(a)} V(a, B)$;
- $V(a, A \supset B) = \bigwedge_{b:a \leq b}^{P(a)} P_{ab}\left(V\left(b, A_{ab}\right) \to^{P(b)} V\left(b, B_{ab}\right)\right)$;
- $V(a, \neg A) = \bigwedge_{b:a \leq b}^{P(a)} P_{ab}\left(V\left(b, A_{ab}\right) \to^{P(b)} \mathbf{0}^{P(b)}\right)$;
- $V(a, \forall x A(x)) = \bigwedge_{b:a \leq b}^{P(a)} \bigwedge_{v \in D(a)}^{P(a)} P_{ab}\left(V\left(b, A_{ab}(v)\right)\right)$;
- $V(a, \exists x A(x)) = \bigvee_{v \in D(a)}^{P(a)} V(a, A(v))$.

Definition 5. *A pair $\langle \mathcal{K}, V\rangle$ is called an algebraic Kripke sheaf model.*

Definition 6. *A formula A is said to be true in an algebraic Kripke sheaf model $\langle \mathcal{K}, V\rangle$ if $V\left(0^{\boldsymbol{M}}, \bar{A}\right) = 1$, this fact is denoted as $\langle \mathcal{K}, V\rangle \models A$. A formula A is said to be valid in an algebraic Kripke sheaf $\mathcal{K}$ if it is true for every valuation V on $\mathcal{K}$.*

3. Heyting-valued structure semantics

The second semantics were originally introduced in [3] and later studied in [1]. **Heyting-valued structure semantics** are defined for (super-)intuitionistic logics with universal and exponential quantifiers and equality as follows:

Definition 7. *Let Ω be a complete Heyting algebra, D a set, and $E : D^2 \to \Omega$ a mapping such that for any $a, b, c \in D$:*

E(1) $E(a, b) = E(b, a)$;
E(2) $E(a, b) \wedge E(b, c) \leq E(a, c)$;
E(3) $\bigvee_{a\in D} E(a, a) = 1$.

A triple $\langle \Omega, D, E\rangle$ is called an Ω-valued structure or a Heyting-valued structure (H.v.s) over Ω. A set D is called an individual domain and its elements are called individuals. A mapping $E : D^2 \to \Omega$ is called a measure of equality.

Let's extend a measure of equality to tuples $\boldsymbol{a}, \boldsymbol{b} \in D^k$ in the following way:

$$E(\boldsymbol{a}, \boldsymbol{b}) := E(a_1, b_1) \wedge \cdots \wedge E(a_k, b_k)$$

and introduce abbreviations:

$$E\boldsymbol{ab} := E(\boldsymbol{a}, \boldsymbol{b}), E\boldsymbol{a} := E\boldsymbol{aa}.$$

Then, by the lemma 4.1.4 from [1] we have:

Proposition 8 ([1])**.** *Every measure of equality E has the following properties:*

(1) $E(a, b) \leq E(a, a)$;
(2) $E\boldsymbol{ab} \leq E\boldsymbol{aa}$;
(3) $E\boldsymbol{ab} \wedge E\boldsymbol{bc} \leq E\boldsymbol{ac}$;
(4) $\bigvee_{a\in D^k} E(\boldsymbol{a}) = 1$.

Definition 9. *A valuation on a H.v.s $F = \langle \Omega, D, E\rangle$ is a map $\varphi : AF_D \to \Omega$ such that for every n-ary predicate P from the language and $\boldsymbol{a}, \boldsymbol{b} \in D^n$ such that $\mathbf{a} =_i \mathbf{b}$ (i.e. $a_k = b_k$ for every $k \neq i$):*

$$\varphi(P(\boldsymbol{a})) \wedge E(a_i, b_i) \leq \varphi(P(\boldsymbol{b})).$$

Definition 10. *Quite naturally we extend the valuation φ to all modal (and intuitionistic) D-sentences (i.e. sentences of the language extended with constants from D) in the following way:*

(1) $\varphi(\perp) := \mathbf{0}$,
(2) $\varphi(a = b) := E(a, b)$;
(3) $\varphi(A \vee B) := \varphi(A) \vee \varphi(B)$;
(4) $\varphi(A \wedge B) := \varphi(A) \wedge \varphi(B)$;
(5) $\varphi(A \supset B) := \varphi(A) \to \varphi(B)$;
(6) $\varphi(\exists x A) := \bigvee_{d\in D}(Ed \wedge \varphi(A(d)))$;
(7) $\varphi(\forall x A) := \bigwedge_{d\in D}(Ed \to \varphi(A(d)))$.

Definition 11. *The pair $\langle F, \varphi\rangle$ of a H.v.s. F and a valuation φ is said to be an algebraic model. A formula A is said to be true in this algebraic model if $\varphi(\bar{A}) = 1$, this fact is denoted as $\langle F, \varphi\rangle \models A$. A formula A is said to be valid in a structure F if it is true in every model over F.*

4. Results

Our research shows that Suzuki Algebraic Kripke Sheaf semantics for the language without equality is reducible to the Heyting-valued structure semantics for a language with equality.

Theorem 12. *For every Suzuki Algebraic Kripke Sheaf $\mathcal{S} = \langle M, D, P\rangle$ and every valuation V on $\mathcal{S}$ there exists a Heyting-valued structure $F = \langle \mathbf{\Omega}, D, E\rangle$ and a valuation φ on F which has the same set of true formulas.*

As the valuation on Algebraic Kripke sheaves is defined in a contravariant manner in order to check the validity of a formula in a model $S := \langle\langle \boldsymbol{M}, D, P\rangle, V\rangle$ we can only look at the valuation at the root point 0^M as the valuation at every other point of the frame can be traced down to the root algebra $P(0^M)$.

An idea of checking the formula valuation only at the root algebra can be encapsulated within the construction of an algebra of monotonically non-decreasing maps from M to $P(0)$. The equivalency between the class of Algebraic Kripke sheaves and the class of Heyting-valued structures over the class of algebras of the above-mentioned type is proven.

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References

[1] D. Gabbay, V. Shehtman, D. Skvortsov. Quantification in Nonclassical Logic. Volume 1. *Elsevier Science*, 2009.
[2] N.-Y. Suzuki. Algebraic Kripke Sheaf Semantics for Non-Classical Predicate Logics. *Studia Logica* 63, pp. 387-416, 1999.
[3] A. Dragalin. On intuitionistic model theory (in Russian). *In Istoriya i metodologiya estestvennyx nauk*, volume XIV, pp. 106–126 MGU, 1973.

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MAKING SENSE OF MIXED CONSEQUENCE RELATIONS

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From the perspective of Susan Haack [7], there is a fundamental worry concerning the proper understanding of many-valued logics. Haack is shows scepticism concerning the intelligibility of truth values other than 'true' and 'false', as it seems to be required in such logics. Still, some of those logics are intelligible from her perspective. Her major claims are: first, in the case of some specific three-valued logics, it may happen that the third truth value represents not a *sui generis* additional value, but rather the *absence* of one of the two truth values 'true' and 'false'. Then, when that is the case, no extra truth value is required, and *two-valuedness* is preserved. Second, and related to the first point, it may happen that the extra truth values are more properly understood, again, not as additional values, but rather as epistemological variants of 'true' and 'false', such as 'true and analytic', or 'true and known to be true'; so, in these scenarios, besides the fact that only two truth values are available, *bivalence* also holds, that is, we preserve the stronger claim that every proposition is either true or false (for a discussion of both cases, see Haack [7, p.213]).

Notice that these worries are spelt out in a context that does not concern the nature of consequence relation, but deals with the very idea of a third truth value and its intelligibility. Haack suggests that two-valuedness and bivalence are the proper ingredients that should be retained if we are to grant understanding and intelligibility to such systems. Her strategy consists in 'explaining' the working of such systems in terms of these more well known concepts, plus some additional semantic or epistemic layer, for instance. In such cases, we don't need a mysterious third truth value. This kind of move allows Haack to make sense of some many-valued logics, with non-mixed consequence relations (cf. [13]).

More recently, there are a number of interesting progresses being made in the literature of mixed consequence relations (cf. [4, 5, 1, 2, 6]). The fact is that although such consequence relations can be defined in general by building on arbitrary many-valued semantics, the more discussed ones are typically defined in the context of some of the three-valued semantics that Haack complained about, namely Strong and Weak Kleene matrices. More importantly, in the case of such discussions of mixed consequence relations, — besides Haackian worries with the intelligibility of the additional truth values —, given that the third value does not enjoy a fixed status if it is a designated value or not, we are exactly in the kind of scenario where it is precisely the understanding of the third truth value that is not always completely clear.

This is, then, another case where intelligibility is threatened — even more than the case with non-mixed consequence relations —, by a mysterious ingredient, the extra truth value, and where a Haackian therapy on such truth values may result in improvements of understanding for those adopting a more traditional view on truth values. The aim of this paper is to do precisely that; we extend Haack's clarificatory strategy for the case of mixed consequence relations, based on Strong and Weak Kleene matrices, and make sense of them from the classicists' perspective.

The paper will be structured as follows. First, we start with Haack's informal ideas on how to interpret many-valued logics. We recap her criticism of extra truth values and her proposed solution in terms of two classical truth values. This will be followed by a presentation of semantic frameworks where the third truth value is substituted by more precise notions incorporating some restrictions on two-valuedness and bivalence (so, the frameworks are either two-valued or bivalent). More specifically, for the *two-valued* semantics, both for Strong and Weak Kleene matrices, we will make use of the plurivalent semantics devised by Graham Priest in [15] and a variation introduced in [11]. Here are some definitions.

Definition 1 (Univalent semantics)**.** *A* univalent semantics *for a propositional language $\mathcal{L}$ is a structure $M{=}\langle \mathcal{V}, \mathcal{D}, \delta\rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and δ contains, for every n-ary connective $*$ in the language, a truth-function $\delta_* : \mathcal{V}^n \to \mathcal{V}$. A* univalent interpretation *is a pair $\langle M, \mu\rangle$, where M is such a structure, and μ is an evaluation function from* Prop *to $\mathcal{V}$. Given an interpretation, μ is extended to a function from* Form *to $\mathcal{V}$ recursively, by the following clause:*

- $\mu(*(A_1,\dots,A_n))=\delta_*(\mu(A_1),\dots,\mu(A_n))$.

Finally, $\Gamma\models_M A$ *iff for all univalent interpretation* $\langle M,\mu\rangle$, *if* $\mu(B)\in\mathcal{D}$ *for all* $B\in\Gamma$, *then* $\mu(A)\in\mathcal{D}$.

Definition 2. *Given a univalent interpretation, the corresponding* general plurivalent interpretation *(gp-interpretation hereafter) is the same, except that we replace* μ *by a one-many evaluation relation,* $\mathfrak{R}$, *between* Prop *and* $\mathcal{V}$. *Given an interpretation,* $\mathfrak{R}$ *is extended to a relation between* Form *and* $\mathcal{V}$ *recursively, by the following clause:*

(‡) $*(A_1,\dots,A_n)\mathfrak{R}v$ *iff for some* $v_1,\dots,v_n$: $[A_i\mathfrak{R}v_i$ *and* $v=\delta_*(v_1,\dots,v_n)]$.

Moreover, we can define two sorts of consequence relation out of this given plurivalent semantics. To this end, we say that $\mathfrak{R}$ tolerantly *designates* A *iff* $A\mathfrak{R}v$ *for some* $v\in\mathcal{D}$, *and* $\mathfrak{R}$ strictly *designates* A *iff for all* $v\in\mathcal{V}$, $A\mathfrak{R}v$ *only if* $v\in\mathcal{D}$ *(iff it is not the case that* $A\mathfrak{R}v$ *for some* $v\notin\mathcal{D}$*). Building on these, we define the corresponding notions of tolerant and strict general plurivalent consequence as follows:*

- $\Gamma\models^M_{g,t}A$ *iff for all gp-interpretations,* $\mathfrak{R}$ *tolerantly designates* A, *if* $\mathfrak{R}$ *tolerantly designates* B, *for all* $B\in\Gamma$.
- $\Gamma\models^M_{g,s}A$ *iff for all gp-interpretations,* $\mathfrak{R}$ *strictly designates* A, *if* $\mathfrak{R}$ *strictly designates* B, *for all* $B\in\Gamma$.

Similarly, the negative plurivalent semantics, proposed in [11], can be obtained as follows.

Definition 3. *Given a univalent interpretation, the corresponding* negative plurivalent interpretation *is the same, except that we replace* μ *by a one-many evaluation relation,* $\mathfrak{R}$, *with the following* negativity condition*:*

- *There is no* $p\in\mathsf{Prop}$ *such that* $p\mathfrak{R}v$ *for all* $v\in\mathcal{V}$.

Given an interpretation, $\mathfrak{R}$ *is extended to a relation between* Form *and* $\mathcal{V}$ *recursively, by the clause* (‡). *The semantic consequence relations* $\models^M_{n,t}$ *and* $\models^M_{n,s}$ *can then be defined as in the general plurivalence case.*

On the other hand, for the *bivalent* semantics, we will make use of other frameworks. That is, for Weak Kleene matrix, we make use of the framework suggested by Hans Herzberger in [8]. More concretely, here are the details.

Definition 4. *A* Herzberger interpretation *of* $\mathcal{L}$ *is a pair* $\langle v_t,v_h\rangle$, *where* $v_t:\mathsf{Prop}\to\{\mathbf{t},\mathbf{f}\}$ *and* $v_h:\mathsf{Prop}\to\{0,1\}$. *Valuations* v_t *and* v_h *are then extended to interpretations* I_t *and* I_h *by the following conditions.*

$I_t(p)=\mathbf{t}$ *iff* $v_t(p)=\mathbf{t}$	$I_h(p)=1$ *iff* $v_h(p)=1$
$I_t(\neg A)=\mathbf{t}$ *iff* $I_t(A)=\mathbf{f}$	$I_h(\neg A)=1$ *iff* $I_h(A)=1$
$I_t(A\wedge B)=\mathbf{t}$ *iff* $I_t(A)=\mathbf{t}$ *&* $I_t(B)=\mathbf{t}$	$I_h(A\wedge B)=1$ *iff* $I_h(A)=1$ *&* $I_h(B)=1$
$I_t(A\vee B)=\mathbf{t}$ *iff* $I_t(A)=\mathbf{t}$ *or* $I_t(B)=\mathbf{t}$	$I_h(A\vee B)=1$ *iff* $I_h(A)=1$ *&* $I_h(B)=1$

Intuitively, the first component v_t represents a truth component and v_h represents an interpretative (e.g., epistemic) dimension. Now that there are four combinations for elements of Prop, we may easily turn the above two-valued semantics into a four-valued semantics, as also observed by Herzberger.[1]

Definition 5. *A four-valued interpretation of* $\mathcal{L}$ *is a function* $v_4:\mathsf{Prop}\to\{\mathbf{t}1,\mathbf{t}0,\mathbf{f}0,\mathbf{f}1\}$. *Given a four-valued interpretation* v_4, *this is extended to a function* I_4 *that assigns every formula a truth value by the following truth functions:*

A	$\neg A$
$\mathbf{t}1$	$\mathbf{f}1$
$\mathbf{t}0$	$\mathbf{f}0$
$\mathbf{f}0$	$\mathbf{t}0$
$\mathbf{f}1$	$\mathbf{t}1$

$A\wedge B$	$\mathbf{t}1$	$\mathbf{t}0$	$\mathbf{f}0$	$\mathbf{f}1$
$\mathbf{t}1$	$\mathbf{t}1$	$\mathbf{t}0$	$\mathbf{f}0$	$\mathbf{f}1$
$\mathbf{t}0$	$\mathbf{t}0$	$\mathbf{t}0$	$\mathbf{f}0$	$\mathbf{f}0$
$\mathbf{f}0$	$\mathbf{f}0$	$\mathbf{f}0$	$\mathbf{f}0$	$\mathbf{f}0$
$\mathbf{f}1$	$\mathbf{f}1$	$\mathbf{f}0$	$\mathbf{f}0$	$\mathbf{f}1$

$A\vee B$	$\mathbf{t}1$	$\mathbf{t}0$	$\mathbf{f}0$	$\mathbf{f}1$
$\mathbf{t}1$	$\mathbf{t}1$	$\mathbf{t}0$	$\mathbf{t}0$	$\mathbf{t}1$
$\mathbf{t}0$	$\mathbf{t}0$	$\mathbf{t}0$	$\mathbf{t}0$	$\mathbf{t}0$
$\mathbf{f}0$	$\mathbf{t}0$	$\mathbf{t}0$	$\mathbf{f}0$	$\mathbf{f}0$
$\mathbf{f}1$	$\mathbf{t}1$	$\mathbf{t}0$	$\mathbf{f}0$	$\mathbf{f}1$

In other words, I_4 is just a direct product of I_t and I_h. Given a many-valued interpretation of the language under consideration, we need to specify the set of designated values to define the semantic consequence relation. To this end, we introduce three different sets of designated values as follows:

$$\mathcal{D}_1:=\{\mathbf{t}1\};\quad \mathcal{D}_2:=\{\mathbf{t}1,\mathbf{t}0\};\quad \mathcal{D}_3:=\{\mathbf{t}1,\mathbf{t}0,\mathbf{f}0\}.$$

We can then define three consequence relations as follows.

[1]Note, however, that Herzberger uses a different notation.

Definition 6. *For $\Gamma\cup\{A\}\subseteq\mathsf{Form}$, and for $i\in\{1,2,3\}$, $\Gamma\models_i A$ iff for all four-valued interpretations v_4, $I_4(A)\in\mathcal{D}_i$ if $I_4(B)\in\mathcal{D}_i$ for all $B\in\Gamma$.*

For Strong Kleene matrix, we make use of not only the framework suggested by Herzberger, along the tweaking suggested by John Martin in [10] (interestingly, Ekaterina Kubyshkina and Dmitry Zaitsev also devised the same semantics with a different motivation in [9] independently of Martin), but also a framework introduced by Matthew Clemens in [3] (cf. [12] for a generalization).[2] The latter semantics is as follows.

Definition 7. *A four-valued interpretation of $\mathcal{L}$ is a function v from Prop to $\{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}$. Given a four-valued interpretation v, this is extended to a function $I:\mathsf{Form}\to\{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}$ as follows:*

	$\neg$
$\langle 1,1\rangle$	$\langle 0,0\rangle$
$\langle 1,0\rangle$	$\langle 0,1\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$
$\langle 0,0\rangle$	$\langle 1,1\rangle$

$\wedge$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$

$\vee$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$
$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$
$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
$\langle 0,1\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,0\rangle$

Note that truth tables for conjunction and disjunction are obtained as a result of adapting min and max definitions, respectively, with the following order on the values: $\langle 0,0\rangle<\langle 0,1\rangle<\langle 1,0\rangle<\langle 1,1\rangle$. Note also that the truth table for negation is very classical.

Finally, by building on these semantic frameworks, we make sense of the mixed consequence relations, by appealing to the more intelligible nature of the truth values and of the resulting semantics involved. As a result, we shall argue, the nature of such consequence relations becomes clear for those having doubts about the meaning of such logics, especially the status of the third truth value and its role in the systems being developed.

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References

[1] E. Barrio, L. Rosenblatt, and D. Tajer. The logics of strict-tolerant logic. *Journal of Philosophical Logic*, 44(5), 551-–571, 2015.

[2] E. Chemlá, and P. Egré. Suszko's problem: Mixed consequence and compositionality. *The Review of Symbolic Logic*, 12(4), 736—767, 2019.

[3] M. Clemens. Ordered pair semantics and negation in LP. *The Australasian Journal of Logic*, 17(5), 201-–205, 2020.

[4] P. Cobreros, P. Egré, D. Ripley, and R. van Rooij. Tolerant, classical, strict. *Journal of Philosophical Logic*, 41(2), 347-385, 2012.

[5] P. Cobreros, P. Egré, D. Ripley, and R. van Rooij. Reaching transparent truth. *Mind*, 122(488), 841–866, 2013.

[6] M. Fitting. Strict/Tolerant Logics Built Using Generalized Weak Kleene Logics. *The Australasian Journal of Logic*, 18(2), 2021.

[7] S. Haack. *Philosophy of Logics*, Cambridge University Press, 1978.

[8] H. G. Herzberger. Dimensions of truth. *Journal of Philosophical Logic*, 2(4), 535-–556, 1973.

[9] E. Kubyshkina, and D. V. Zaitsev. Rational agency from a truth-functional perspective. *Logic and Logical Philosophy*, 25(4), 499–520, 2016.

[10] J. N. Martin. An axiomatization of Herzberger's 2-dimensional presuppositional semantics. *Notre Dame Journal of Formal Logic*, 18(3), 378–382, 1977.

[11] H. Omori. Halldéen's Logic of Nonsense and its expansions in view of Logics of Formal Inconsistency. *Proceedings of DEXA 2016*, 129–133, 2016.

[12] H. Omori and J. R. B. Arenhart. A Generalization of Ordered-Pair Semantics. *International Workshop on Logic, Rationality and Interaction*, 149–157, 2021.

[13] H. Omori and J. R. B. Arenhart. Haack meets Herzberger and Priest. *2022 IEEE 52st International Symposium on Multiple-Valued Logic (ISMVL)*, 2022.

[14] H. Omori and J. R. B. Arenhart. Change of logic, without change of meaning. *Theoria*, 2023.

[15] G. Priest. Plurivalent Logics. *The Australasian Journal of Logic*, 11(1), 1–13, 2014.

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COMPLETENESS OF THE LOGIC HC IN A SPECIAL SPACE

ANASTASIIA ONOPRIENKO

In a commentary to his collected works [1], Kolmogorov remarked that his paper [2] "was written in hope that with time, the logic of solution of problems [i.e., intuitionistic logic] will become a permanent part of a [standard] course of logic. A unified logical apparatus was intended to be created, which would deal with objects of two types — propositions and problems." Melikhov [3] constructed a formal system QHC (a joint logic of problems and propositions) with two types of variables: problem and proposition. Formulas of QHC are built from variables by using standart classical and intuitionistic connectives $\wedge, \vee, \neg, \rightarrow$ (including classical and intuitionistic falsity constants 0 and $\perp$ respectively), modalities ! and ? and quantifiers $\forall, \exists$. We do not differentiate graphically between standart classical and intuitionistic connectives ($\wedge, \vee, \neg, \rightarrow$, except classical and intuitionistic falsity constants), since they can be distinguished by the type of the formulas that they act upon.

All propopositions (all problems) satisfy all inference rules and axioms schemes of classical (intuitionistic) predicate logic. Formulas of this two types are interconnected by two operators. The modality ! inputs a proposition p and outputs a problem $!p$ with intended reading "Find a proof of p". The modality ? inputs a problem α and outputs a proposition $?\alpha$ with intended reading "There exists a solution of α". There are following axioms schemes and inference rules for modalities:

(1) $!(p \rightarrow q) \rightarrow (!p \rightarrow !q)$;
(2) $?(\alpha \rightarrow \beta) \rightarrow (?\alpha \rightarrow ?\beta)$;
(3) $\dfrac{p}{!p}$;
(4) $\dfrac{\alpha}{?\alpha}$;
(5) $p \rightarrow ?!p$;
(6) $\alpha \rightarrow !?\alpha$;
(7) $\neg!0$.

Melikhov examined several types of topological models for logic QHC [4], but completeness fails even for propositional part HC of this logic. The author considered algebraical models and Kripke-type semantic for HC. For this types of models the author proved completeness theorem and finite model property [5]. In [6] we constructed Kripke semantics of the QHC logic and demonstrated that the completeness theorem (and even the strong completeness theorem) of the QHC logic is valid for the given class of models.

In [7] we propose the topological models of the HC logic. Let us describe these models. Suppose that (X, τ) is an arbitrary nonempty topological space and A is its dense subset. The triplet (X, τ, A) is named by topological frame. The variables p of the proposition sort are interpreted as arbitrary subsets $|p| \subset A$. The variables α of the problem sort are interpreted as arbitrary open subsets $|\alpha| \subset X$. The truth values of the formulas are determined by induction on the construction of the formula; furthermore, the truth value of any formula of the proposition sort is some subset A and the truth value of any formula of the problem sort is an open subset X. The classical and intuitionistic connectives and constants are interpreted in the following standard manner (see, e.g., [8]).

The classical connectives and the constant 0:
$|p \wedge q| = |p| \cap |q|$;
$|p \vee q| = |p| \cup |q|$;
$|p \rightarrow q| = (A \setminus |p|) \cup |q|$;
$|0| = \varnothing$.

The intuitionistic connectives and the constant $\perp$:
$|\alpha \wedge \beta| = |\alpha| \cap |\beta|$;
$|\alpha \vee \beta| = |\alpha| \cup |\beta|$;
$|\alpha \rightarrow \beta| = \mathrm{Int}((X \setminus |\alpha|) \cup |\beta|)$;
$|\perp| = \varnothing$.

The modalities are interpreted as follows:

$|?\alpha| = A \cap |\alpha|$;

$|!p| = X \setminus \mathrm{Cl}(A \setminus |p|)$.

In other words, $|!p|$ is a union of all open sets U such that $U \cap A \subset |p|$.

As usual, a formula p of the proposition sort (a formula α of the problem sort) is true in this topological model of the HC if $|p| = A$ ($|\alpha| = X$).

The following theorem holds [7].

Theorem 1 (correctness and completeness)**.**
1) If a formula is deducible in HC*, then it is true in any topological model of the logic* HC.
2) If a formula φ is non deducible in the logic HC*, then there exists a topological model of the logic* HC *in which the formula φ is not true.*

It is clear that the topological models of the HC is an enrichment of the standart topological models of the intuitionistic logic. Melikhov proved [3] that the HC is a conservative extension of the intuitionistic logic and the logic S4 (where modality $\Box$ of the logic S4 is understood as the derived modality ?! of the HC). The following interesting fact is valid. If we put a dense subset A coinciding with the whole set X and consider just the classical part of the HC with this derived modality, then we obtain the standart topological models of the logic S4 (see, e.g.[9]).

Let us clarify the formulation of the theorem 1 as follows.

Theorem 2.
1) HC *is the logic of the class of all finite topological spaces with a dense subset.*
2) HC *has the effective finite model property with respect to the class of topological spaces with a dense subset.*

It is known that S4 is the logic of the class of all finite topological spaces. There are lots of classical results on topological models of the logic S4 with much more mathematical content. One of them is McKinsey and Tarski's theorem: S4 is the logic of any dense-in-itself metric separable space [10]. Another results concern the completeness of the logic S4 in special spaces: the Cantor space $\mathbb{C}$, the rational line $\mathbb{Q}$ and the real line $\mathbb{R}$ [11, 12]. One can try to find out whether similar results are valid for HC. First of all we need the following theorem [5].

Definition 3. *Audit set frame is a triplet* $(W, \preccurlyeq, \mathrm{Aud})$*, where* $(W, \preccurlyeq)$ *is a standard intuitionistic frame (W is a nonempty set, $\preccurlyeq$ is a partial order),* $\mathrm{Aud} \subset W$ *is a confinal subset of audit states (i.e.* $\forall a \in W\ \exists b \in \mathrm{Aud}\ a \preccurlyeq b$*).*

Audit set frame are introduced by Artemov and Protopopescu as semantics of intuitionistic epistemic logic IEL$^+$ [13].

Let us define the evaluation $\models$ for intuitionistic formulas by standard way, for classical formulas by naturally way only in audit states, and for modalities by this way:

$$a \models ?\alpha \Leftrightarrow a \models \alpha \text{ (for } a \in \mathrm{Aud}),$$
$$a \models !p \Leftrightarrow \forall b \in \mathrm{Aud}(a \preccurlyeq b \Rightarrow b \models p) \text{ (for } a \in W).$$

We obtain an audit set model of HC.

Theorem 4. *Logic* HC *is the logic of the class of all audit set model with $W = T_2$ — the infinite binary tree.*

One can strengthen the theorem 3.

Theorem 5. *Logic* HC *is the logic of the class of all audit set model with $W = T_2$ in which* Aud *and* $T_2 \setminus \mathrm{Aud}$ *both are confinal.*

Using this fact we prove the following result.

Theorem 6. *Logic* HC *is complete with respect to topological frame* $(\mathbb{A}, \tau_{\leqslant}, \mathbb{Q})$*, where $\mathbb{A}$ is the set of algebraic numbers, $\tau_{\leqslant}$ is the standart order topology and $\mathbb{Q}$ is the set of rational numbers.*

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References

[1] A. N. Kolmogorov. On the papers on intuitionistic logic, Selected Works of A. N. Kolmogorov. Vol. I, Mathematics and its Applications (Soviet Series), vol. 25, Kluwer, Dordrecht, 1991, pp. 451.
[2] A. N. Kolmogoroff. Zur Deutung der intuitionistischen Logik, Math. Z. 35 (1932), no. 1, 58–65.
[3] S. A. Melikhov. A Galois connection between classical and intuitionistic logics. I: Syntax. arXiv:1312.2575, 2013.
[4] S. A. Melikhov. A Galois connection between classical and intuitionistic logics. II: Semantics. arXiv:1504.03379, 2015.
[5] A. A. Onoprienko. Kripke semantics for the logic of problems and propositions, Sb. Math., 211:5 (2020), 709–732.
[6] A. A. Onoprienko. The predicate version of the joint logic of problems and propositions, Sb. Math., 213:7 (2022), 981–1003.
[7] A. A. Onoprienko. Topological Models of Propositional Logic of Problems and Propositions, Moscow University Mathematics Bulletin, 77:5 (2022), 236-241.
[8] H. Rassiowa, R. Sikorski. The Mathematics of Metamathematics, Monografie Matematyczne, Vol.41 (Panstwowe Wydawnictwo Naukowe, Warszawa, 1963).
[9] J. C. C. McKinsey. A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology, J. Symbolic Logic 6 (1941), 117–124.
[10] J. McKinsey, A. Tarski. The algebra of topology, Annals of Mathematics, 45 (1944), 141–191.
[11] M. Aiello, J. van Benthem, G. Bezhanishvili. Reasoning about space: The modal way. J. Logic Comput., 13:6 (2003), 889–920.
[12] J. van Benthem, G. Bezhanishvili, B. ten Cate, D. Sarenac. Modal logics for products of topologies. Studia Logica, 84:3 (2005),369–392.
[13] S. Artemov, T. Protopopescu. Intuitionistic Epistemic Logic. arXiv:1406.1582v2, 2014.

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TEMPORAL EPISTEMIC LOGIC FOR REASONING WITH DELAY IN AWARENESS

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The present talk is based on joint work with V. V. Dolgorukov.

Standard epistemic logic fails to represent the real notion of reasoning as it suffers from the logical omniscience problem. This problem arises because of distribution axiom:

$$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi) \tag{1}$$

This axiom with a combination of necessitation rule for operator K_i guarantees that knowledge is closed under logical inference. Therefore, we get over-idealized picture of reasoning which allows to model only ideal agents. However, there is a way out, which is associated with the weakening of the concept of knowledge. The solution includes the division of knowledge into implicit and explicit one. While the implicit knowledge remains closed under logical inference, the explicit knowledge does not. This approach is used in awareness logics.

Firstly, this solution was proposed for the notion of belief [6], then it was considered also for knowledge [7]. Implicit knowledge refers to the knowledge that the agent possesses, but does not realise it. In other words, the agent is not aware of this knowledge. Explicit knowledge is defined as knowledge with awareness.

Class of awareness logics can be divided into static [6] and dynamic ones. The latter takes into account the dynamics of information and are better suited for formalizing agent interaction scenarios. There are dynamic awareness logics that are constructed in the DEL framework [8, 1], temporal frameworks [3, 2], and rule-based frameworks [5, 4]. All the dynamic approaches consider epistemic changes as something external. In other words, there is no any formal "trigger" in these systems that initiates the acquisition of awareness and explicit knowledge. I would like to propose a logic that explains why agents become aware and obtain explicit knowledge.

I will focus on the internal dynamic of awareness that includes changes occurring in the agent state of knowledge. My approach involves internal changes in implicit agents' knowledge that cause obtaining of awareness and explicit knowledge. I will present temporal epistemic logic for awareness with minimum delay (TELAMD). TELAMD formalizes the deliberation of imperfect agents in a constantly changing flow of information. In contrast with other dynamic awareness logics, it has one special axiom that plays a role of internal trigger for awareness and so explicit knowledge.

Let us assume a fixed countable set $Prop$ of atomic propositions and a fixed finite set Ag of agents.

Definition 1 (Syntax)**.** *Formulas of the language $\mathcal{L}$ are given by the Backus-Naur form*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid A_i\varphi \mid X\varphi \mid Y\varphi$$

with $p \in Prop$, $i \in Ag$.

Other logical connectives and constants are introduced as standard abbreviations. Formula $K_i\varphi$ is read as "the agent i knows φ implicitly", $A_i\varphi$ is read as "the agent i is aware of φ". Formula "$\Box_i\varphi$" is interpreted as "agent i knows explicitly φ". It is defined as a syntactic abbreviation for the following combinations of implicit knowledge and awareness:

$$\Box_i\varphi := K_i\varphi \wedge A_i\varphi$$

Definition 2 (TELAMD Model)**.** *A model of temporal epistemic model for agents with delay in awareness is a tuple*

$$\mathcal{M} = (W, (\sim_i)_{i\in Ag}, \rightsquigarrow, A, V)$$

where

- *$W \neq \varnothing$ is a set of possible worlds*
- *$\sim_i \subseteq W \times W$ is an accessibility relation for an agent i*
- *$\rightsquigarrow \subseteq W \times W$ is a temporal accessibility relation for the epistemic evolution of the possible worlds*
- *$A : Ag \times W \to \mathcal{P}(\mathcal{L})$ is awareness function*

- *$V : Prop \to \mathcal{P}(W)$ is an evaluation function for propositional variables*

A relation $\sim_i$ is an equivalence relation. The relation "$x \rightsquigarrow y$" is interpreted as "y is the next temporal stage of the possible world x".

Definition 3 (Restrictions on TELAMD Model)**.** *Evolutionary relation $\rightsquigarrow$ has the following properties*

(1) $\forall x \forall y \forall z ((y \rightsquigarrow x \wedge z \rightsquigarrow x) \Rightarrow y = z)$
reverse functionality
(2) $\forall x \forall y \forall z ((x \rightsquigarrow y \wedge y \sim_i z) \Rightarrow \exists w (w \rightsquigarrow z \wedge x \sim_i w))$
perfect recall
(3) $\forall x \forall y ((x \sim_i y \wedge \exists z (x \rightsquigarrow z)) \Rightarrow \exists w (y \rightsquigarrow w))$
knowledge about the absence of the forward dead end
(4) $\forall x \forall y ((x \sim_i y \wedge \neg \exists z (x \rightsquigarrow z)) \Rightarrow \neg \exists w (y \rightsquigarrow w))$
knowledge about the forward dead end
(5) $\forall x \forall y ((x \sim_i y \wedge \exists z (z \rightsquigarrow x)) \Rightarrow \exists w (w \rightsquigarrow y))$
knowledge about the absence of the backward dead end
(6) $\forall x \forall y ((x \sim_i y \wedge \neg \exists z (z \rightsquigarrow x)) \Rightarrow \neg \exists w (w \rightsquigarrow y))$
knowledge about the backward dead end.

Property (1) claims that evolutionary relation $\rightsquigarrow$ is reverse functional. Restriction (2) corresponds to "perfect recall" property. Restrictions (3)–(6) claim that backward and forward temporal branching of different length is forbidden.

Definition 4 (Semantics)**.** *The truth of a modal formula φ in a pointed model $(\mathcal{M}, w)$ is defined as follows:*

- $\mathcal{M}, w \vDash p \iff w \in V(p)$
- $\mathcal{M}, w \vDash \neg\varphi \iff \mathcal{M}, w \nvDash \varphi$
- $\mathcal{M}, w \vDash \varphi \wedge \psi \iff \mathcal{M}, w \vDash \varphi$ *and* $\mathcal{M}, w \vDash \psi$
- $\mathcal{M}, w \vDash K_i\varphi \iff \forall w'(w \sim_i w' \Rightarrow \mathcal{M}, w' \vDash \varphi)$
- $\mathcal{M}, w \vDash X\varphi \iff \forall w'(w \rightsquigarrow w' \Rightarrow \mathcal{M}, w' \vDash \varphi)$
- $\mathcal{M}, w \vDash Y\varphi \iff \forall w'(w' \rightsquigarrow w \Rightarrow \mathcal{M}, w' \vDash \varphi)$
- $\mathcal{M}, w \vDash A_i\varphi \iff \varphi \in A_i(w)$

The axiom system $\mathcal{TELAMD}$ for temporal epistemic logic with minimal delay in awareness consists of the following axiom schemes and inference rules.

(1) all instances of classical tautologies
(2) axioms $S5$ for operator K_i
(3) $X(\varphi \to \psi) \to (X\varphi \to X\psi)$
(4) $Y(\varphi \to \psi) \to (Y\varphi \to Y\psi)$
(5) $\varphi \to X\hat{Y}\varphi$
(6) $\varphi \to Y\hat{X}\varphi$
(7) $\hat{Y}\varphi \to Y\varphi$
(8) $K_iX\varphi \to XK_i\varphi$
(9) $X\bot \to K_iX\bot$
(10) $\neg X\bot \to K_i\neg X\bot$
(11) $Y\bot \to K_iY\bot$
(12) $A_i\varphi \to XA_i\varphi$
(13) $A_i\varphi \to K_iA_i\varphi$
(14) $(\hat{Y}\neg K_i\varphi \wedge K_i\varphi) \to XA_i\hat{Y}\varphi$

and inference rules (modus ponens and necessitation rule for operators K_i, X, Y):

MP $\dfrac{\varphi, \varphi \to \psi}{\psi}$

Nec_{K_i} $\dfrac{\vdash \varphi}{\vdash K_i\varphi}$

Nec_X $\dfrac{\vdash \varphi}{\vdash X\varphi}$

Nec_Y $\dfrac{\vdash \varphi}{\vdash Y\varphi}$

Axiom 14 is the awareness trigger. It expresses the main idea of the internal dynamics of awareness. If the agent did not know something implicitly, and then knows it implicitly, so at the next step they would become aware of it.

Theorem 5 (Soundness and Strong Completeness of TELAMD)**.** *Logic TELAMD is sound and strongly complete w.r.t. TELAMD models. Then,*

$$\Gamma \vDash \varphi \Longleftrightarrow \Gamma \vdash \varphi \tag{2}$$

Theorem 6 (Decidability of TELAMD)**.** *Logic TELAMD is decidable.*

The complexity of the satisfiability problem for TELAMD is an open problem.

References

[1] J. van Benthem, F. R. Velázquez-Quesada. The dynamics of awareness. *Synthese*, 177, 5–27, 2010.

[2] H. N. Duc. Reasoning about Rational, but not Logically Omniscient, Agents *Journal of Logic and Computation*, 7(5), 633–648, 1997.

[3] R. Fagin, J. Y. Halpern. Belief, Awareness, and Limited Reasoning. *Artificial Intelligence*, 34(1), 39–76, 1988.

[4] J. Y. Halpern, R. Pucella. Dealing with logical omniscience: Expressiveness and pragmatics. *Artificial Intelligence*, 175(1), 220–235, 2011.

[5] M. Jago. Epistemic Logic for Rule-Based Agents. *Journal of Logic, Language and Information*, 18(1), 131–158, 2009.

[6] H. J. Levesque. A logic of implicit and explicit belief. *AAAI'84: Proceedings of the Fourth AAAI Conference on Artificial Intelligence. Austin: AAAI Press*, 198–202, 1984.

[7] M. Y. Vardi. On epistemic logic and logical omniscience. *Proceedings of the 1986 conference. San Francisco: Morgan Kaufmann Publishers Inc*, 293–305, 1986.

[8] F. R. Velázquez-Quesada. Inference and update. *Synthese*, 169(2), 283–300, 2009.

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ALGEBRAIC SEMANTICS FOR HYPERGRAPH LAMBEK CALCULUS

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1. Introduction

The Lambek calculus L is a non-commutative substructural logic designed to model the syntax of natural languages. Formulas of the Lambek calculus are built from primitive ones using the product $\cdot$ and two divisions $\backslash$, $/$. Since the seminal work of Joachim Lambek [4], numerous extensions and variants of L have been studied, for example, the nonassociative Lambek calculus NL, the commutative Lambek calculus LP, the multimodal Lambek calculus NL$\diamondsuit$, the displacement calculus D.

In [6, 7], we proposed a generalization of L called the hypergraph Lambek calculus HL. This calculus has a complex hypergraph syntax: for example, the inductive definition of a formula of HL says that, if we take a graph G such that its edges are labeled by already defined formulas, then $\times(G)$ is a formula as well. The operation $\times$ generalizes the product of L; HL also includes the hypergraph division $\div$. It turns out that HL is a very general framework such that one is able to embed L, NL, LP, NL$\diamondsuit$, D in it by modeling sequents of each of these calculi with certain kinds of hypergraphs. Moreover, many fundamental properties of L and of related calculi can be straightforwardly generalized to HL, e.g. the cut elimination theorem. Thus establishing facts about HL could be useful to highlight some general properties of the Lambek calculus and of its modifications.

The Lambek calculus has several semantics, in particular, the algebraic semantics: L is the logic of residuated semigroups [3, p. 101]. A residuated semigroup is a structure of the form $\langle M, \circ, \leq, \backslash, / \rangle$ such that $\langle M, \circ \rangle$ is a semigroup, $\langle M, \leq \rangle$ is a partially ordered set, $\circ$ is monotonic w.r.t. $\leq$, and for all $a, b, c \in M$ it holds that $a \circ b \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a \backslash c$. The completeness theorem for L w.r.t. this semantics is proved using canonical models. Another semantics for L is the language semantics [5], which corresponds to linguistic applications of L.

In this work, we develop an algebraic semantics for HL by defining the notion of a residuated hypergraph semigroup. The definition itself is of most interest to us since it adapts the basic algebraic concept of a semigroup operation for the hypergraph setting. While a standard semigroup operation $a \circ b$, roughly speaking, combines a and b placed in a row one after another, a hypergraph semigroup has a family of operations represented as arbitrary hypergraphs labeled by elements of this semigroup. It is not hard to prove that HL is sound and complete w.r.t. residuated hypergraph semigroups: the completeness proof is done by the canonical model construction as well.

2. Preliminaries

2.1. **Hypergraphs.** A *ranked set* M is the set along with a *rank function* $rk : M \to \mathbb{N}$; let M_n denote $\{a \in M \mid rk(a) = n\}$. Given a ranked set of *labels* $\mathcal{T}$, a *hypergraph* G over $\mathcal{T}$ is a tuple $G = \langle V_G, E_G, att_G, lab_G, ext_G \rangle$ where V_G is a finite set of *nodes*, E_G is a finite set of *hyperedges*, $att_G : E_G \to V_G^*$ assigns a string (i.e. an ordered multiset) of *attachment nodes* to each hyperedge, $lab_G : E_G \to \mathcal{T}$ labels each hyperedge by some element of $\mathcal{T}$ in such a way that $rk(lab_G(e)) = |att_G(e)|$ whenever $e \in E_G$, and $ext_G \in V_G^*$ is a string of *external nodes*. Hypergraphs are always considered up to isomorphism. The set of all hypergraphs with labels from $\mathcal{T}$ is denoted by $\mathcal{H}(\mathcal{T})$. The *rank function* rk_G (or rk, if G is clear) is defined as follows: $rk_G(e) := |att_G(e)|$. Besides, $rk(G) := |ext_G|$.

Let $[m] = \{1, \ldots, m\}$ and $[0] = \emptyset$. A *handle* $a^\bullet$ is a hypergraph $\langle [n], [1], att, lab, 1 \ldots n \rangle$ where $att(1) = 1 \ldots n$ and $lab(1) = a$ ($a \in \mathcal{T}$, $rk(a) = n$). *A string graph* $\mathrm{sg}(w)$ *induced by the string* $w = a_1 \ldots a_n$ is a hypergraph of the form $\langle \{v_i\}_{i=0}^n, [n], att, lab, v_0 v_n \rangle$ where $att(i) = v_{i-1} v_i$, $lab(i) = a_i$.

Given a hypergraph $H \in \mathcal{H}(\mathcal{C})$ and a function $f : \mathcal{C} \to \mathcal{T}$, *a relabeling* H^f is the hypergraph $H^f = \langle V_H, E_H, att_H, f \circ lab_H, ext_H \rangle$. It is required that $rk(a) = rk(f(a))$ for $a \in \mathcal{C}$.

2.2. **Hyperedge replacement.** The *replacement of a hyperedge* e_0 *in* G *(*$e_0 \in E_G$*) by a hypergraph* H (such that $rk(e_0) = rk(H)$) is done as follows:

(1) remove e_0 from E_G;
(2) insert an isomorphic copy of H (H and G must consist of disjoint sets of nodes and hyperedges);

(3) for each $i = 1, \dots, rk(e_0)$, fuse the i-th external node of H with the i-th attachment node of e_0 (formally, the set of new nodes is $(V_G \sqcup V_H)/{\equiv}$ where $\equiv$ is the smallest equivalence relation satisfying $att_G(e_0)(i) \equiv ext_H(i)$).

The result is denoted as $G[e_0/H]$. If several hyperedges of a hypergraph are replaced by other hypergraphs, then the result does not depend on the order of the replacements; moreover the result does not change, if replacements are done simultaneously [2]. If $e_1, \dots, e_k$ are distinct hyperedges of a hypergraph H and they are replaced by hypergraphs $H_1, \dots, H_k$ resp., then the result is denoted $H[e_1/H_1, \dots, e_k/H_k]$.

2.3. **Hypergraph Lambek Calculus.** To define formulas (called *types*) of HL, we fix a ranked set Pr of *primitive types*; we require that for each $k \in \mathbb{N}$ there are infinitely many $p \in Pr$ such that $rk(p) = k$. Besides, we fix special labels $\$_n, n \in \mathbb{N}$ ("placeholders") and set $rk(\$_n) = n$; let us agree that these labels do not belong to any other set considered in the definition of the calculus or in the definitions from the next section. If $\mathcal{T}$ is a ranked set of labels, then $\mathcal{H}^{\$}(\mathcal{T})$ denotes the set of hypergraphs such that, for each $G \in \mathcal{H}^{\$}(\mathcal{T})$, labels of all hyperedges of G, except for one, are from $\mathcal{T}$, and the remaining one equals $\$_d$ for some d. The hyperedge of $G \in \mathcal{H}^{\$}(\mathcal{T})$ labeled by $\$_d$ is denoted by $e^{\$}_G$.

The ranked set of *types* Tp is defined inductively as follows:

(1) All primitive types are types.
(2) Let $N \in Tp$ be a type, and let D be a hypergraph from $\mathcal{H}^{\$}(Tp)$ (i.e. its labels, except for one, are already defined types, and the remaining one is $\$_d$) $\underline{\text{such that } |E_D| > 1}$. Let also $rk(N) = rk(D)$. Then $N \div D$ is also a type such that $rk(N \div D) := rk(e^{\$}_D)$.
(3) If $M \in \mathcal{H}(Tp)$ is a hypergraph labeled by already defined types $\underline{\text{such that } |E_M| > 0}$, then $\times(M)$ is also a type, and $rk(\times(M)) := rk(M)$.

A *hypergraph sequent* is a structure of the form $H \to A$ where $H \in \mathcal{H}(Tp)$ is a hypergraph labeled by types $\underline{\text{such that } |E_H| > 0}$ and A is a type such that $rk(H) = rk(A)$.

The hypergraph Lambek calculus HL derives hypergraph sequents. The only axiom of HL is of the form $A^\bullet \to A$ where $A \in Tp$ ($A^\bullet$ is a handle). There are four inference rules of HL:

$$\frac{H[e/N^\bullet] \to A \quad H_1 \to lab_D(d_1) \quad \dots \quad H_k \to lab_D(d_k)}{H\left[e/D[e^{\$}_D/(N \div D)^\bullet, d_1/H_1, \dots, d_k/H_k]\right] \to A}\ (\div L) \qquad \frac{D[e^{\$}_D/F] \to N}{F \to N \div D}\ (\div R)$$

$$\frac{H[e/M] \to A}{H[e/(\times(M))^\bullet] \to A}\ (\times L) \qquad \frac{H_1 \to lab_M(m_1) \quad \dots \quad H_l \to lab_M(m_l)}{M[m_1/H_1, \dots, m_l/H_l] \to \times(M)}\ (\times R)$$

Here $N \div D$, $\times(M) \in Tp$ are types; $e \in E_H$; $E_D = \{e^{\$}_D, d_1, \dots, d_k\}$, $E_M = \{m_1, \dots, m_l\}$. Note that $H[e/a^\bullet]$ is simply the result of relabeling of the hyperedge e by a.

The underlined parts of the definitions correspond to the so-called Lambek's restriction, which says that antecedents of sequents of L must be non-empty. If these parts are removed, then we come up with *the hypergraph Lambek calculus allowing edgeless premises* HL*. Note that Lambek's restriction in HL as defined in this work differs from that from [6, 7], and we claim that the new one is more appropriate (although we are not going to discuss this in the current work).

3. Algebraic Semantics for HL

By $\mathcal{H}^+(\mathcal{T})$ we denote the set of hypergraphs over $\mathcal{T}$ that contain at least one hyperedge.

Definition 1. *A* hypergraph semigroup *is a structure $\langle A, Op \rangle$ where*

- *A is a ranked set (in other words, a hypergraph semigroup is ω-sorted);*
- *$Op : \mathcal{H}^+(A) \to A$ is a function such that $rk(Op(H)) = rk(H)$.*

The function Op must satisfy the following properties:

(1) *(handle identity) $Op(a^\bullet) = a$ for all $a \in A$;*
(2) *(associativity) $Op\left(H\left[e/(Op(G))^\bullet\right]\right) = Op(H[e/G])$ for all appropriate G, H.*

Definition 2. *A* partially ordered hypergraph semigroup *is a hypergraph semigroup $\langle A, Op, \leq \rangle$ equipped with a family of partial orders $\leq\ =\ \{\leq_n \subseteq A_n \times A_n\}_{n=0}^{\infty}$ defined on A_n for each n such that:*

$$a \leq_n b \Rightarrow Op(H[e/a^\bullet]) \leq_{rk(H)} Op(H[e/b^\bullet]), \quad \textit{for each } H \in \mathcal{H}^+(A) \textit{ and } e \in E_H \textit{ with } rk(e) = n.$$

Definition 3. *A* residuated hypergraph semigroup (RHS) *is a partially ordered hypergraph semigroup $\langle A, Op, \leq, Res \rangle$ where the function Res is defined on pairs (a, D) such that $a \in A$, $D \in \mathcal{H}^{\$}(A)$, $|E_D| > 1$, $rk(a) = rk(D)$, and it satisfies the following properties:*

(1) *$Res(a, D) \in A$ and $rk(Res(a, D)) = rk(e^{\$}_D)$;*
(2) *(residual property) $Op(D[e^{\$}_D/b^\bullet]) \leq_{rk(D)} a \Leftrightarrow b \leq_{rk(e^{\$}_D)} Res(a, D)$.*

Informally, if we understand $a \circ b$ in a semigroup as a combination of a and b according to $\circ$, then $Op(H)$ can be understood as a combination H of elements of a hypergraph semigroup that are labels of hyperedges of H. In particular, if H is a string graph $\mathrm{sg}(ab)$, then elements a and b are combined in a standard, linear way, and we return to the notion of a semigroup, as the following example shows:

Example 1. Given an RHS $\mathcal{A} = \langle A, Op, \leq, Res\rangle$, let us consider the structure $Str_{\mathcal{A}} = \langle A_2, \circ, \leq_2, \backslash, /\rangle$ where $a \circ b := Op(\mathrm{sg}(ab))$, $b\backslash a := Res(a, \mathrm{sg}(b\,\$_2))$, $a/b := Res(a, \mathrm{sg}(\$_2\, b))$. Then $Str_{\mathcal{A}}$ is a residuated semigroup: for example, associativity holds because $(a\circ b)\circ c = Op\left(\mathrm{sg}\left(Op\left(\mathrm{sg}\left(ab\right)\right)c\right)\right) = Op\left(\mathrm{sg}\left(abc\right)\right) = Op\left(\mathrm{sg}\left(a\, Op\left(\mathrm{sg}\left(bc\right)\right)\right)\right) = a \circ (b \circ c)$. It is an interesting question whether the converse holds as well: given a residuated semigroup $\mathcal{S}$, is there an RHS $\mathcal{A}$ such that $Str_{\mathcal{A}}$ is isomorphic to $\mathcal{S}$?

An example of an RHS generalizing the residuated semigroup of languages is the following one:

Example 2. Given a ranked alphabet Σ, the structure $\mathcal{HLAN}^+_\Sigma := \langle A_\Sigma, Op, \leq, Res\rangle$ is an RHS where

- A_Σ consists of hypergraph languages, i.e. $L \in A_\Sigma$ iff for some $n \in \mathbb{N}$ it holds that $L \subseteq \{H \in \mathcal{H}^+(\Sigma) \mid rk(H) = n\}$; then we say that $rk(L) = n$. Formally, we have to consider ω distinct copies of the empty language $\emptyset_n$ such that $rk(\emptyset_n) = n$ (in order to avoid many-sortedness of $\emptyset$);
- $Op(H) = \{H[e_1/G_1] \dots [e_m/G_m] \mid 1 \leq i \leq m, G_i \in lab_H(e_i)\}$ where $H \in \mathcal{H}^+(A_\Sigma)$ and $E_H = \{e_1, \dots, e_m\}$;
- $Res(L, D) = \{G \in \mathcal{H}^+(\Sigma) \mid \forall D_1 \in lab_D(d_1) \dots \forall D_l \in lab_D(d_l)\ D[e^{\$}_D/G, d_1/D_1, \dots, d_l/D_l] \in L\}$ where $D \in \mathcal{H}^{\$}(A_\Sigma)$, $|E_D| > 1$ and $E_D = \{e^{\$}_D, d_1, \dots, d_l\}$;
- $L_1 \leq_n L_2$ iff $L_1 \subseteq L_2$.

Definition 4. *An RHS-model $(\mathcal{A}, v)$ of* HL *is an RHS $\mathcal{A} = \langle A, Op, \leq, Res\rangle$ along with a valuation $v : Tp \to A$ such that:*
(1) *$rk(v(T)) = rk(T)$;*
(2) *$v(\times(M)) = Op(M^v)$;*
(3) *$v(N \div D) = Res(v(N), D^v)$; note that for convenience we also define $v(\$_n) = \$_n$.*

Definition 5. *A sequent $H \to C$ is valid in an RHS-model $(\mathcal{A}, v)$ iff $v(\times(H)) \leq_n v(C)$ ($n = rk(C)$).*

Theorem 6. *A sequent is derivable in* HL *if and only if it is valid in all RHS-models.*

The proof is essentially the same as for L, and it uses the standard canonical model construction.

If the hypergraph counterpart of Lambek's restriction is removed everywhere (i.e. if we replace $\mathcal{H}^+(A)$ by $\mathcal{H}(A)$ in the algebraic definitions and remove the requirement that $|E_D| > 1$ in Definition 3), then we come up with the notion of *residuated hypergraph monoid*, and we can prove soundness and completeness of HL* w.r.t. the class of such models. The term *monoid* can be explained as follows: if hypergraphs without hyperedges are allowed, then, in Example 1, $Str_{\mathcal{A}}$ is a residuated monoid with the unit $e = Op(\mathrm{sg}(\Lambda))$ where Λ is the empty word; note that $\mathrm{sg}(\Lambda)$ is the hypergraph $\langle v_0, \emptyset, \emptyset, \emptyset, v_0v_0\rangle$ with zero hyperedges and one node.

Concerning the language semantics for HL, in [6], we prove that the product-free fragment of HL is complete w.r.t. models defined on $\mathcal{HLAN}^+_\Sigma$ (the proof is standard, it generalizes that from [1]). The question whether HL with product is complete w.r.t. hypergraph language models remains open.

4. Conclusion

The hypergraph Lambek calculus HL is a general framework extending several variants of the Lambek calculus. The calculus HL is convenient for representing non-linear structures, for example, syntax trees used in linguistics or sequents of the displacement calculus. In our opinion, its generality is the reason why studying it might be important: this helps us to find some common properties of different calculi based on the Lambek calculus and studied in the literature as well as to see where their similarities end. This, in particular, is applicable to model-theoretic questions: the algebraic semantics introduced in this work shows us a way to define an algebraic semantics for each particular fragment of HL. Moreover, the definition of RHS gives us a fresh point of view at the basic notion of a semigroup operation.

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References

[1] W. Buszkowski. Compatibility of a categorial grammar with an associated category system. *Zeitschr. Math. Log. Grundl. Math.* 28, 229–238, 1982. 10.1002/malq.19820281407

[2] F. Drewes, H.-J. Kreowski, A. Habel. Hyperedge Replacement Graph Grammars. Handbook of Graph Grammars and Computing by Graph Transformations, Volume 1: Foundations. *World Scientific*, 95-162, 1997. 10.1142/9789812384720_0002.

[3] Editor(s): N. Galatos, P. Jipsen, T. Kowalski, H. Ono. Chapter 2 – Substructural Logics and Residuated Lattices. In: Residuated Lattices: An Algebraic Glimpse at Substructural Logics, *Studies in Logic and the Foundations of Mathematics, Elsevier*, 151, 75-139, 2007. 10.1016/S0049-237X(07)80007-3.

[4] J. Lambek. The Mathematics of Sentence Structure. *The American Mathematical Monthly*, 65(3), 154–170, 1958. DOI: 10.1080/00029890.1958.11989160

[5] M. Pentus. Models for the Lambek Calculus. *Ann. Pure Appl. Log.*, 75(1-2), 179–213, 1995. 10.1016/0168-0072(94)00063-9.

[6] T. Pshenitsyn. Grammars Based on a Logic of Hypergraph Languages. Berthold Hoffmann and Mark Minas, editors: Proceedings Twelfth International Workshop on Graph Computational Models, GCM@STAF 2021, Online. *EPTCS*, 350, 1–18, 2021. 10.4204/EPTCS.350.1

[7] T. Pshenitsyn. Hypergraph Lambek Grammars. *Journal of Logical and Algebraic Methods in Programming*, 129, 100798, 2022. 10.1016/j.jlamp.2022.100798

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MEZHIROV'S GAME FOR INTUITIONISTIC LOGIC AND ITS VARIATIONS

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Game semantics allows us to look at basic logical concepts from another side. This approach to logic has a long history, there are plenty of different types of games: provability games, semantic games, etc [10,11]. And there is an interesting type of provability games called Mezhirov's game proposed by Iliya Mezhirov for intuitionistic logic of propositions (IPC) and Grzegorczyk modal logic (Grz) [1,2]. This idea was developed in many different directions; for example, in 2008 in the joint paper with N. Vereschagin a game semantics was given for affine and linear logic [3]. Independently G. Japaridze worked on game semantics for linear logic [4]. Mezhirov's games for minimal propositional logic (MPC), logic of functional frames ($KD!$) and logic of serial frames (KD) were introduced in 2021 by A. Pavlova [5].

Mezhirov's game semantics for intuitionistic logic is interesting because of its simplicity and strong connection with Kripke semantics and Kripke models. The game between Opponent and Proponent starts with a formula φ. And Proponent has a winning strategy iff φ is an intuitionistic tautology. The connection between the game and Kripke models manifests itself in building strategy for Opponent from a Kripke model (Opponent "walks" from one world of a model to another) and in the reconstruction of a model from Opponent's winning strategy (in which there exists a world where φ is false). And these procedures are connected to each other.

In my study, I try to generalize Mezhirov's result in two directions: to generalize to intuitionistic logic of predicates (introduce a game between Opponent and Proponent with at least the same connection with Kripke models or with special classes of them) and to the case of a connection not only between the game and tautologies of logic ($\vDash \varphi$), but also between the game and entailment from infinite sets of formulas ($T \vDash \varphi$).

The purpose of building such game was to get a theorem of kind "Proponent has a winning strategy in $\mathcal{C}_0$ iff $\mathcal{O}_0 \vDash \varphi$", where $\vDash$ is the semantic consequence defined by some class of predicate Kripke frames. I initially thought about just logic of all Kripke models, i.e. it would be a game for intuitionistic logic of predicates directly. But it turned out that in such case some fundamental problems arise and it is natural to expand the logic (to use a smaller class of Kripke frames). Moreover, description of such variations (not just logic of all Kripke models) could be useful, since, in general, Kripke semantics for superintuitionistic predicate logic is rather weak (e.g. [9]). And I managed to get a description (based on the game I built) for several variations. So let me describe the rules of the game.

Let Ω be the elementary intuitionistic language (without function symbols; language will contain $\perp$, and the set of logical connectives will be $\{\to, \wedge, \vee\}$, where $\neg A$ will be considered as $A \to \perp$), and we will use Kripke models for intuitionistic logic of predicates [6,7] (I will call sets of constants in each world "individual domains" (or "the set of objects") and use symbol Δ). For the set of formulas Γ and set of objects (constants) Δ let $\mathcal{F}(\Gamma, \Delta) = \{P[c_1, ..., c_n] | P[x_1, ..., x_n]$ is a subformula of some formula from Γ and free variables of it are only $x_1, ..., x_n; c_i \in \Delta\}$ (so $\mathcal{F}$ in some ways is a set of all "subformulas" of formulas from Γ). Players Opponent and Proponent will be associated with their sets $\mathcal{O}$ and $\mathcal{P}$. The position in the game is a trio $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \Delta)$. In each position $\mathcal{C}$: $\mathcal{O}$ and $\mathcal{P}$ are subsets of $\mathcal{F}(\Gamma, \Delta)$, where Δ is taken from $\mathcal{C}$ (and can only expand in the game process) and Γ is fixed at the beginning of the game and does not change until the end and equals to $\mathcal{O}_0 \cup \{\varphi\}$ (where $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ is a starting position; Δ_0 is an exact set of all constants contained in formulas from Γ). Proponent moves by adding new formulas from $\mathcal{F}$ to $\mathcal{P}$, Opponent moves by expanding Δ (he can add nothing to Δ if he wants; and he can add to Δ not just constants from Ω) and than adding new formulas from $\mathcal{F}$ to $\mathcal{O}$.

The only thing left to define is who must move in a position $\mathcal{C}$. To do that, let us firstly define the notion of truth relation $\Vdash$ in $\mathcal{C}$ for formulas from $\mathcal{F}(\Gamma, \Delta)$:

$$\mathcal{C} \nVdash \bot$$
$$\mathcal{C} \Vdash A[c_1, ..., c_n] \rightleftharpoons A[c_1, ..., c_n] \in \mathcal{O}$$
$$\mathcal{C} \Vdash \varphi \star \psi \rightleftharpoons \varphi \star \psi \in \mathcal{O} \cup \mathcal{P} \text{ and } (\mathcal{C} \Vdash \varphi) \star (\mathcal{C} \Vdash \psi),\ \star \in \{\to, \wedge, \vee\}$$
$$\mathcal{C} \Vdash qxP[x] \rightleftharpoons qxP[x] \in \mathcal{O} \cup \mathcal{P} \text{ and } q\alpha \in \Delta(\mathcal{C} \Vdash P[\alpha]),\ q \in \{\exists, \forall\}$$

where A is a predicate symbol, $arity(A) = n$, $c_i \in \Delta$, P - formula with only one free variable. A star in the case of $(\mathcal{C} \Vdash \varphi) \star (\mathcal{C} \Vdash \psi)$ means logical meta connective and behaves like a classical connective (the same for q in $q\alpha \in \Delta$).

Let us call a formula from $\mathcal{P}$ Proponent's mistake if it is false in the current position (the same for $\mathcal{O}$ and Opponent). If Opponent has no mistakes but Proponent has, then Proponent moves. Otherwise, Opponent must move. And if after a turn of a fixed player he must move again, he loses. If the game goes on infinitely (each player manages to pass a turn to the other player each turn), Proponent wins; also let us call formulas from $\mathcal{O} \cup \mathcal{P}$ marked formulas.

Now let us consider several examples of the game. In the first game $\mathcal{C}_0 = (\varnothing, \{\varphi\}, \varnothing)$, where $\varphi = \forall y \exists x (P[x] \to P[y])$. Because Δ is empty, there are no formulas in $\mathcal{F}$ of the form $\exists x(P[x] \to P[c])$, so Proponent has no mistakes, it's Opponent's turn. It is enough for him to just expand Δ, and it will be Proponent's turn. Proponent takes all formulas of the kind $\exists x(P[x] \to P[c])$ and $P[c] \to P[c]$ and passes turn to Opponent. He will do the same (expand Δ) and the game goes on infinitely.

In the second game $\mathcal{C}_0 = (\varnothing, \{\varphi\}, \{c\})$, where $\varphi = \neg P[c] \to \neg \exists x P[x]$. φ is an implication, both sending and conclusion of it is not marked, therefore false in the current position. So φ is true, it's Opponent's turn. He expand Δ to $\{c, \alpha\}$ and add to $\mathcal{O}$ formulas $\neg P[c]$, $\exists x P[x]$, $P[\alpha]$. He might not add $\exists x P[x]$ to $\mathcal{O}$ and turn would still be passed to Proponent. But in this case Proponent would have an opportunity to add to $\mathcal{P}$ $\neg \exists x P[x]$ and make this formula true in position (because $\exists x P[x]$ would not be marked), and Opponent still would have needed to add $\exists x P[x]$. After that, Proponent will not be able to pass turn to Opponent, therefore, he will lose.

In the third game let $\mathcal{C}_0 = (\varnothing, \{\varphi\}, \varnothing)$, where $\varphi = \forall x[(P[x] \to \forall x P[x]) \to \forall x P[x]] \to \forall x P[x]$ (*Casari's schema or Casari's formula* [8]). Again φ is an implication, it's Opponent's turn. He needs to make sending false, so he expand Δ and add to $\mathcal{O}$ all formulas $(P[\alpha] \to \forall x P[x]) \to \forall x P[x]$ and sending of the φ: $\forall x[(P[x] \to \forall x P[x]) \to \forall x P[x]]$. Than Propopent creates mistakes for Opponent by adding to $\mathcal{P}$ all formulas $(P[\alpha] \to \forall x P[x])$. To get rid of mistakes, Opponent needs to add all $P[\alpha]$, and than Proponent just add to $\mathcal{P}$ $\forall x P[x]$. The only thing Opponent can do now is to expand Δ and repeat everything again. As we can see, this is the winning strategy for Proponent, but φ is not true in all Kripke models. This formula will give us a useful class of Kripke frames (class of all Kripke frames in which Casari's formula is valid; let us call it Casari's class (Kripke frame is from Casari's class iff in every countable sequence of worlds ω_i their individual domains Δ_i remain finite and stabilize; so class of Casari's Kripke frames includes all Noetherian Kripke frames)).

It seems to me that, informally, this game (and Mezhirov's game for propositional intuitionistic logic) could be understood as follows: Opponent is trying to build a theory that belies Proponent's assertion that ϕ follows from $\mathcal{O}_0$ (or, in the case of $\mathcal{O}_0 = \varnothing$, is trying to build a theory that shows that Proponent's thesis (φ) is not valid in general). And this theory must be coherent (Opponent must have no mistakes), otherwise his approach is considered unsuccessful.

While getting closer to results, I should mention that there was some interest in considering variation of the game with only finite Δ (and Opponent can expand Δ adding only finite number of objects) because of better connection with Kripke models, so there appeared results for two variations of the game (described one (let us call it infinite) and the same but with finite Δ (finite variation)).

Theorem 1. *In the infinite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with possibly infinite* $\mathcal{O}_0$*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all Noetherian Kripke frames.*

Theorem 2. *In the infinite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with only finite* $\mathcal{O}_0$*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all Casari's Kripke frames.*

These theorems lead, inter alia, to the fact that logics of Noetherian Kripke frames and of Casari's Kripke frames have the same weak entailment. Similar results we can see for the finite variation.

Theorem 3. *In the finite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with possibly infinite* $\mathcal{O}_0$*, but with only finite* Δ_0*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all Noetherian Kripke frames with only finite individual domains* Δ *in each world.*

Theorem 4. *In the finite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with only finite* $\mathcal{O}_0$*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all Casari's Kripke frames with only finite individual domains* Δ *in each world.*

Theorem 5. *In the finite variation, Proponent has a winning strategy in position* $\mathcal{C}_0 = (\mathcal{O}_0, \{\varphi\}, \Delta_0)$ *(with only finite* $\mathcal{O}_0$*) iff* $\mathcal{O}_0 \vDash \varphi$*, where* $\vDash$ *is the entailment in logic of all finite Kripke frames with only finite individual domains* Δ *in each world.*

As we can see, in the case of only finite individual domains in each world, logics of Noetherian Kripke frames, of Casari's Kripke frames and of finite Kripke frames have the same weak entailment.

As I mentioned, the main goal of this study was to find a game with strong connection with Kripke models. Partially, this has been achieved (proofs for all 5 theorems contain building a strategy for Opponent by "walking" from one world of a model to another); in addition, some connections have been established between weak entailment of logics of some classes. But the next step would be to find a trio: a class of Kripke frames, a game semantics and a calculus (probably, an infinitary sequent calculus) with the same strong entailment. In this case, it is better to take a simpler class of Kripke frames in terms of the possible calculus for this class. So, because of this, Casari's class looks better than the Noetherian class. Therefore, I am trying right now to change rules of the game to get the same strong entailment as in logic of all Kripke frames from Casari's class.

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References

[1] I. Mezhirov. Game semantics for intuitionistic and modal (Grz) logic. Master's thesis, Lomonosov Moscow State University, Department of Mechanics and Mathematics. 2006.
[2] I. Mezhirov. A Game Semantics for Grz. Journal of Logic and Computation, Volume 16, Issue 5, October 2006, Pages 663-669.
[3] I. Mezhirov, N. Vereshchagin. On Game Semantics of the Affine and Intuitionistic Logics. Part of the Lecture Notes in Computer Science book series (LNAI,volume 5110). 2008.
[4] G. Japaridze. In the beginning was game semantics. In: Games: Unifying Logic, Language and Philosophy. O. Majer, A.-V. Pietarinen and T. Tulenheimo, eds. Springer 2009, pp.249-350.
[5] A. Pavlova. Provability Games for Non-classical Logics. In: Silva, A., Wassermann, R., de Queiroz, R. (eds) Logic, Language, Information, and Computation. WoLLIC 2021. Lecture Notes in Computer Science, vol 13038. 2021.
[6] D. van Dalen. Intuitionistic Logic. In Handbook of Philosophical Logic. 2nd ed., edited by D. M. Gabbay and F. Guenthner, 1–114. Dordrecht, Netherlands: Kluwer Academic, 2002.
[7] A. Dragalin. Mathematical Intuitionism: Introduction to Proof Theory. Providence, RI: American Mathematical Society, 1988.
[8] L. Esakia. Quantification in intuitionistic logic with provability smack. pp.26-28, 1998.
[9] D. Skvortsov. An Incompleteness Result for Predicate Extensions of Intermediate Propositional Logics. Conference: Advances in Modal Logic 4, papers from the fourth conference on "Advances in Modal logic". 2002.
[10] J. van Benthem. Logic Games: From Tools to Models of Interaction, pp. 183–216, March 2011.
[11] J. van Benthem. Logic in Games. The MIT Press, Cambridge. 2014.

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TOPOLOGICAL SEMANTICS OF THE PREDICATE MODAL CALCULUS QGL EXTENDED WITH NON-WELL-FOUNDED PROOFS

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The Gödel-Löb provability logic GL is a well-known propositional unimodal system. According to its arithmetical interpretation, the modal connective $\Box$ corresponds to the standard provability predicate *"... is provable in Peano arithmetic* PA*"*. As shown by Solovay [7], a formula is a theorem of GL if and only if every its arithmetical translation is a theorem of PA. In other words, GL captures the properties of formal provability of PA that are provable in PA itself.

The Gödel-Löb provability logic GL can be also described by means of its relational semantics. This logic is complete with respect to the class of irreflexive transitive Kripke frames without infinite ascending chains. However, GL is only weakly complete for its relational interpretation.

Strong completeness is achieved if one considers topological (or neighbourhood) semantics of the given system. The class of topological spaces corresponding to GL consists of all scattered topological spaces (X, τ), where the modal connective $\Box$ is interpreted as the co-derived-set operator $cd_\tau(Y) = \{x \in X \mid \exists U \in \tau\, (x \in U \wedge U \setminus \{x\} \subset Y)\}$.

An interesting feature of GL is that this system allows cyclic and non-well-founded reasoning. In [3, 1], it was shown that GL can be defined by means of a sequent calculus allowing non–well-founded proofs. In [4, 5], the standard axiomatic calculus for GL was extended with non–well-founded derivations and various topological completeness results for the obtained system were established.

In the present talk, we focus on a first-order predicate version of GL denoted by QGL. We consider this system in a language without function symbols and constants and define it by the following axioms and inference rules.

Axioms:

- tautologies of classical propositional logic,
- $\Box(A \to B) \to (\Box A \to \Box B)$,
- $\Box A \to \Box\Box A$,
- $\Box(\Box A \to A) \to \Box A$,
- $\forall x\, A(x) \to A(y)$,
- $\forall x\, (A \to B) \to (A \to \forall x\, B)$, where $x \notin FV(A)$.

Inference rules:

$$\mathsf{mp}\,\frac{A \qquad A \to B}{B}\,, \qquad \mathsf{nec}\,\frac{A}{\Box A}\,, \qquad \mathsf{gen}\,\frac{A}{\forall x\, A}\,.$$

Thanks to Montagna [2], we know that QGL is not arithmetically complete. He also showed that this system is not complete with respect to its Kripke semantics. However, it is not something out of the ordinary. In many cases, predicate versions of Kripke complete modal propositional systems are incomplete for their relational interpretations. Whether QGL is topologically complete, we do not know, but we conjecture that it is not.

We introduce an extension of QGL obtained by allowing non-well-derivations in the QGL calculus. A *non-well-founded derivation, or ∞-derivation,* is a (possibly infinite) tree whose nodes are marked by predicate modal formulas and that is constructed according to the rules (mp), (gen) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec). Below is an example of an ∞-derivation:

$$\mathsf{mp}\,\dfrac{\mathsf{nec}\,\dfrac{\mathsf{gen}\,\dfrac{\mathsf{mp}\,\dfrac{\begin{matrix}\vdots\\ \Box\forall x_2\, P_2(x_2)\end{matrix} \qquad \Box\forall x_2\, P_2(x_2) \to P_1(x_1)}{P_1(x_1)}}{\forall x_1\, P_1(x_1)}}{\Box\forall x_1\, P_1(x_1)} \qquad \Box\forall x_1\, P_1(x_1) \to P_0(x_0)}{P_0(x_0)}\,.$$

A *non-well-founded proof, or ∞-proof,* is an ∞-derivation, where all leaves are marked by axioms of QGL. We write $\mathsf{QGL}_\infty \vdash A$ if there is an ∞-proof with the root marked by A.

Our main result is that QGL_∞ is complete with respect to the class of predicate topological frames for QGL_∞ with constant domains. We define a *predicate topological frame for* QGL_∞ as a tuple (X, τ, D), where (X, τ) is a scattered topological space and D is a non-empty domain. Note that, in the case of topological semantics, the constant domain condition does not imply validity of the Barcan formula in contrast to the case of relational frames.

Let us recall some basic notions of semantics of predicate modal systems. A *valuation in* D is a function sending each n-ary predicate letter to an n-ary relation on D, and a *variable assignment* is a function from the set of variables $\mathit{Var} = \{x_0, x_1, x_2, \dots\}$ to the domain D. For QGL_∞, a *predicate topological model* $\mathcal{M} = (X, \tau, D, \xi)$ is a predicate topological frame (X, τ, D) of QGL_∞ together with an indexed family of valuations $\xi = (\xi_w)_{w \in X}$ in D. Elements of the set X are usually called *worlds* of the model.

The *truth of a formula* A *at a world* w *of a model* $\mathcal{M}$ *under a variable assignment* h is defined as

- $\mathcal{M}, w, h \nvDash \bot$,
- $\mathcal{M}, w, h \vDash P(x_1, \dots, x_n) \iff (h(x_1), \dots, h(x_n)) \in \xi_w(P)$,
- $\mathcal{M}, w, h \vDash A \to B \iff \mathcal{M}, w, h \nvDash A$ or $\mathcal{M}, w, h \vDash B$,
- $\mathcal{M}, w, h \vDash \Box A \iff \exists U \in \tau\ (w \in U$ and $\forall w' \in U \setminus \{w\}\ \mathcal{M}, w', h \vDash A)$,
- $\mathcal{M}, w, h \vDash \forall x\, A \iff \mathcal{M}, w, h' \vDash A$ for any varible assignment h' such that $h' \stackrel{x}{=} h$,

where $h' \stackrel{x}{=} h$ means that $h'(y) = h(y)$ for each $y \in \mathit{Var} \setminus \{x\}$.

A *formula* A *is true in* $\mathcal{M}$ if A is true at all worlds of $\mathcal{M}$ under all variable assignments. In addition, A *is valid in a frame* $\mathcal{F}$ if A is true in all models over $\mathcal{F}$.

Theorem 1 (topological completeness)**.** *For any formula A, $\mathsf{QGL}_\infty \vdash A$ if and only if A is valid in every predicate topological frame of QGL_∞.*

In order to obtain this result, we focus on a proof-theoretic presentation of QGL_∞ in a form of a sequent calculus allowing non–well-founded proofs. This calculus is defined by the following initial sequents and inference rules:

$$\Gamma, P(\vec{x}) \Rightarrow P(\vec{x}), \Delta, \qquad \Gamma, \bot \Rightarrow \Delta,$$

$$\to_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}, \qquad \to_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta},$$

$$\forall_{\mathsf{L}} \frac{\Gamma, A(y), \forall x\, A \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta}, \qquad \forall_{\mathsf{R}} \frac{\Gamma \Rightarrow A(y), \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}\ (y \notin FV(\Gamma \cup \Delta)),$$

$$\Box\ \frac{\Gamma, \Box\Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A, \Delta}.$$

In addition, every infinite branch in a non-well-founded proof of this calculus must contain infinitely many applications of the rule ($\Box$).

Our proof of Theorem 1 is inspired by two other sequent-based completeness proofs. We follow the proof for classical predicate logic based on reduction trees and the proof for the system GL extended with non-well-founded derivations from [5].

We also establish a strong version of Theorem 1. We write $\Gamma \vDash A$ if for any predicate topological model $\mathcal{M} = (X, \tau, D, \xi)$, any world w of $\mathcal{M}$ and any variable assignment $h\colon \mathit{Var} \to D$

$$\forall B \in \Gamma\ \mathcal{M}, w, h \vDash B \Longrightarrow \mathcal{M}, w, h \vDash A.$$

Combining the standard ultaproduct costruction and Shehtman's ultrabouquet construction from [6], we obtain the following proposition.

Proposition 2 (compactness)**.** *If $\Gamma \vDash A$, then there is a finite subset Γ_0 of Γ such that $\Gamma_0 \vDash A$.*

Applying the previous results, we prove strong completeness for the following syntactic consequence relation. We put $\Gamma \vdash A$ if there is an ∞-derivation δ with the root marked by A such that, for each leaf a of δ that is not marked by an axiom, a is marked by a formula from Γ, and there are no applications of the rules (gen) and (nec) on the path from the root of δ to the leaf a.

Corollary 3 (strong completeness). *For any set of formulas* Γ *and any formula* A*,*

$$\Gamma \vdash A \Longleftrightarrow \Gamma \vDash A.$$

In conclusion, we note one interesting question: what is the complexity of QGL_∞? It may well turn out that this system is not computably enumerable. In the area of predicate provability logic, there is an example. According to Vardanian's result, the set of predicate modal formulas provable in PA under any interpretation is Π^0_2-complete [8, 9].

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References

[1] R. Iemhoff, *Reasoning in circles*, Liber Amicorum Alberti. A Tribute to Albert Visser (Jan van Eijck et al., ed.), College Publications, London, 2016, pp. 165–178.

[2] F. Montagne, *The predicate modal logic of provability*, Notre Dame Journal of Formal Logic **25** (1987), 179–189.

[3] D. Shamkanov, *Circular proofs for the Gödel-Löb provability logic*, Mathematical Notes **96** (2014), no. 3, 575–585.

[4] ———, *Global neighbourhood completeness of the Gödel-Löb provability logic*, Logic, Language, Information, and Computation. 24th International Workshop, WoLLIC 2017 (London, UK, July 18-21, 2017) (Juliette Kennedy and Ruy de Queiroz, eds.), Lecture Notes in Computer Science, no. 103888, Springer, 2017, pp. 358—-371.

[5] ———, *Non-well-founded derivations in the Gödel-Löb provability logic*, Rev. Symb. Log. **13** (2020), no. 4, 776–796.

[6] V. Shehtman, *On neighbourhood semantics thirty years later*, We Will Show Them! Essays in Honour of Dov Gabbay (S. Artemov et al., ed.), vol. 2, College Publications, London, 2005, pp. 663–692.

[7] R. Solovay, *Provability interpretations of modal logic*, Israel Journal of Mathematics **25** (1976), 287–304.

[8] V. Vardanyan, *Arithmetic comlexity of predicate logics of provability and their fragments*, Sov. Math. Dokl **33** (1986), no. 3, 569–572.

[9] A. Visser and M. de Jonge, *No escape from Vardanyan's theorem*, Arch. Math. Logic **45** (2006), no. 5, 539–554.

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WHAT DO 'EVIDENCE' AND 'TRUTH' MEAN IN THE LOGICS OF EVIDENCE AND TRUTH

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Logics of evidence and truth (LETs) are paraconsistent and paracomplete logics that extend the logic of first-degree entailment (FDE), also known as Belnap-Dunn logic (see e.g. [4, 10]). LETs are further developments of the work on paraconsistency carried out by da Costa, Carnielli, and Coniglio, among others (see e.g. [7, 9, 11]), and they have been conceived to formalize the deductive behavior of positive and negative evidence, which can be either conclusive or non-conclusive. LETs are equipped with a classicality operator $\circ$ that recovers classical negation for sentences in its scope:

$A, \neg A \nvdash B$, for some A and B, but for every A and B, $\circ A, A, \neg A \vdash B$,
$B \nvdash A \vee \neg A$, for some A and B, but for every A and B, $\circ A, B, \vdash A \vee \neg A$,

LETs thus divide the sentences of the language into two groups: one subjected to classical logic, the other subjected to a subclassical logic that is FDE or an extension of FDE.

It is well-known that FDE can be interpreted in terms of preservation of information, and LETs can also be interpreted along the same lines. Accordingly, a formula $\circ A$ means that the information conveyed by A, either positive or negative, is reliable, and it is assumed that reliable information is subjected to classical logic. Thus, whenever $\circ A$ holds, A cannot be contradictory (on pain of triviality).

The motivation for adding the connective $\circ$ to FDE is to be able to represent, in addition to the four scenarios expressed by FDE, two more scenarios that specifically concern reliable information. More precisely, when $\circ A$ does not hold, and so the information conveyed by A is not known to be reliable, we have the four scenarios of FDE. But when $\circ A$ holds, these four scenarios are narrowed down to just two: either the information A or the information $\neg A$ is reliable, but not both. The resulting six scenarios are expressed by a two-valued non-deterministic semantics as follows:

No reliable information A: $v(\circ A) = 0$:

1. Only positive information A: $v(A) = 1$, $v(\neg A) = 0$;
2. Only negative information A: $v(A) = 0$, $v(\neg A) = 1$;
3. No positive nor negative information A: $v(A) = 0$, $v(\neg A) = 0$;
4. Both positive and negative information A: $v(A) = 1$, $v(\neg A) = 1$.

Either A or $\neg A$ is reliable: $v(\circ A) = 1$:

5. Reliable information A: $v(A) = 1$, $v(\neg A) = 0$;
6. Reliable information $\neg A$: $v(A) = 0$, $v(\neg A) = 1$.

The logic LET_F^- is a sort of minimal logic of evidence and truth obtained by extending FDE with the following rules:

$$\frac{\circ A \quad A \quad \neg A}{B}\ EXP^\circ \qquad\qquad \frac{\begin{matrix}A\\ \vdots\\ B\end{matrix} \quad \begin{matrix}\neg A\\ \vdots\\ B\end{matrix}}{B}\ PEM^\circ$$

The connective $\circ$ indicates the presence of reliable information, thus it allows distinguishing circumstances in which only positive (negative), though non-reliable, information A is available (viz., $v(A) = 1$ and $v(\circ A) = 0$) from circumstances in which there is positive (negative) reliable information A (viz., $v(A) = v(\circ A) = 1$). In other words, unlike in FDE, in LET_F^- (as well as in the other LETs) we are able to distinguish, among the six scenarios above, 1 from 5, as well as 2 from 6.

In [6], [14], and [5] sentential LETs that extend, respectively, Nelson's logic $N4$ [1], FDE and $FDE^\to$ (FDE added with a classical implication, see [12]), along with sound and complete two-valued non-deterministic semantics (see [11]), have been presented. In [2], Kripke models for sentential LETs have been investigated. First-order LETs with two-valued semantics and Kripke semantics with variable domains have been investigated in [3] and [13]. In [8] six-valued non-deterministic semantics for some LETs have been presented, as well as extensions of LETs with rules for propagation of classicality have been investigated.

In a broad sense, propagation of classicality is how classical behavior propagates from less complex to more complex sentences, and vice-versa. The LETs investigated so far enjoy the following property: Let $\mathbf{L} \in \{LET_F^-, LET_J, LET_K, LET_F\}$. Let $\Gamma = \{\circ\neg^{n_1} A_1, \ldots, \circ\neg^{n_m} A_m\}$, for $n_i \geq 0$ (where $\neg^{n_i}$, $n_i \geq 0$, represents n_i occurrences of negations before the formula A_i). Then, for any formula B formed with $A_1, \ldots, A_m$ over the signature $\{\neg, \wedge, \vee, \rightarrow\}$ (and $\{\neg, \wedge, \vee\}$ in the case of LET_F and LET_F^-), $\Gamma \vdash_{\mathbf{L}} B \vee \neg B$, and $\Gamma, B, \neg B \vdash_{\mathbf{L}} C$, that is, B behaves classically in this context.

Proof. This result is proved for LET_F in [14, Fact 31], and for LET_J in [2, Proposition 7]. Proofs for LET_F^- and LET_K can be obtained similarly. □

Although the classical behavior is transmitted from less complex to more complex formulas, the classicality operator $\circ$ is not; that is, the inferences $\circ p \vdash \circ\neg p$ and $\circ p, \circ q \vdash \circ(p \# q)$ ($\# \in \{\vee, \wedge\}$), for example, do not hold. The rules above, EXP° and PEM° indicates this fact, since they are a sort of 'elimination rules' for $\circ$, and there is no the respective introduction rule. However, in order to rigorously express the idea of dividing the sentences of the language into two groups, which is an essential point of LETs (as well as LFIs and da Costa's C_n hierarchy) it would be desirable for the classicality operator $\circ$ to be transmitted as well.

Recall that $\circ A \wedge A$ and $\circ A \wedge \neg A$ are intended to mean that the information conveyed, respectively, by A and by $\neg A$, is considered reliable. In addition, positive and negative reliable information behaves like truth and falsity in classical logic. As a consequence, from $\circ A \wedge A$ one should be able to infer that $A \vee B$ is also reliable for any B, no matter whether $\circ B$ holds or not, and so $\circ(A \vee B) \wedge (A \vee B)$ holds. For if this were not the case, that is, if both $(A \vee B)$ and $\neg(A \vee B)$ held, $\circ A$ could not hold, since $\neg(A \vee B)$ implies $\neg A$. On the other hand, from $\circ A \wedge \neg A$, it cannot be inferred that $\neg(A \vee B)$ is reliable, because to conclude $\neg(A \vee B)$ both $\neg A$ and $\neg B$ are required. Hence, from $\circ A \wedge \neg A$, $\circ(A \vee B)$ cannot be inferred. This suggests the validity of the following inferences:

(1) $\circ A, A \vdash \circ(A \vee B) \wedge (A \vee B)$,
(2) $\circ B, B \vdash \circ(A \vee B) \wedge (A \vee B)$,
(3) $\circ A, \neg A, \circ B, \neg B \vdash \circ(A \vee B) \wedge \neg(A \vee B)$,

and so on. The point of these inferences is that positive (resp. negative) reliable information behaves like truth (resp. falsity) in classical logic: in order to have a $A \vee B$ false, we need both A and B false, but A true is enough to conclude $A \vee B$ true.

When we extend the logic LET_F^- with rules of propagation of classicality, we obtain a finitely valued logic, dubbed LET_F^+, with six semantic values that correspond to the six scenarios mentioned above.

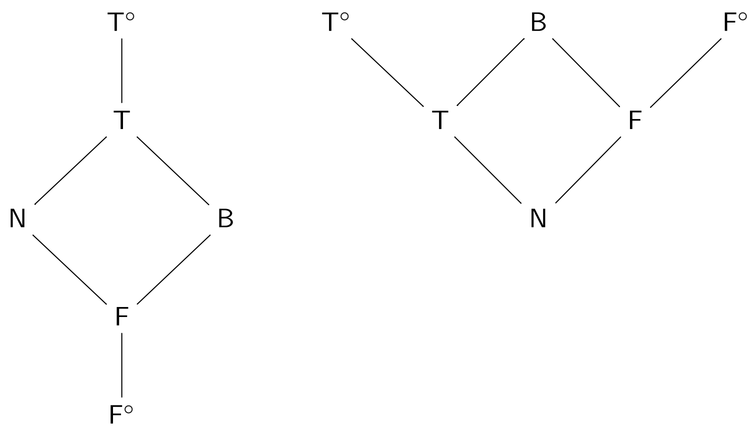
No reliable information A:

1. Only positive information A: $v(A) = \mathsf{T}$,
2. Only negative information A: $v(A) = \mathsf{F}$,
3. No positive nor negative information A: $v(A) = \mathsf{N}$,
4. Both positive and negative information A: $v(A) = \mathsf{B}$,

Either A or $\neg A$ is reliable:

5. Reliable positive information A: $v(A) = \mathsf{T}^\circ$,
6. Reliable negative information A: $v(A) = \mathsf{F}^\circ$.

The six-valued semantics sketched above is based on twist-structures, and they define the lattice *L6* and the meet semi-lattice *A6* that extend the logical lattice *L4* and the approximation *A4*, defined by the four values of FDE (see [4]), with the two additional values *conclusively true* (T°) and *conclusively false* (F°):



The aim of this talk is twofold. First, we present and discuss some recent developments in LETs, in particular the logic LET_F^+. Second, in order to answer some criticisms raised against LETs, we explain the notions of evidence and information underlying the intuitive interpretation of LETs, and also clarify in which sense the deductive behavior of conclusive evidence is expressed in terms of preservation of truth.

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References

[1] A. Almukdad and D. Nelson. Constructible falsity and inexact predicates. *The Journal of Symbolic Logic*, 49(1): 231–233, 1984.

[2] H. Antunes, W. Carnielli, A. Kapsner, and A. Rodrigues. Kripke-style models for logics of evidence and truth. *Axioms*, 9(3), 2020.

[3] H. Antunes, A. Rodrigues, W. Carnielli, and M.E. Coniglio. Valuation semantics for first-order logics of evidence and truth. *Journal of Philosophical Logic*, 2022. doi: 10.1007/s10992-022-09662-8

[4] N.D. Belnap. How a computer should think. In G. Ryle, editor, *Contemporary Aspects of Philosophy.* Oriel Press, 1977a. Reprinted in *New Essays on Belnap-Dunn Logic*, Springer, 2019.

[5] W. Carnielli and A. Rodrigues. On the philosophy and mathematics of the logics of formal inconsistency. In J.-Y. Beziau et al., editor, *New Directions in Paraconsistent Logic – Springer Proceedings in Mathematics & Statistics 152*, pages 57–88. Springer India, 2015b.

[6] W. Carnielli and A. Rodrigues. An epistemic approach to paraconsistency: a logic of evidence and truth. *Synthese*, 196:3789–3813, 2017. URL https://rdcu.be/ctJRQ.

[7] W. Carnielli, M.E. Coniglio, and J. Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 14, pages 1–93, Amsterdam, 2007. Springer-Verlag.

[8] M. Coniglio and A. Rodrigues. On six-valued logics of evidence and truth expanding Belnap-Dunn four-valued logic. arXiv:2209.12337, 2022.

[9] N. da Costa. *Sistemas Formais Inconsistentes.* Curitiba: Editora da UFPR (1993), 1963.

[10] J.M. Dunn. Intuitive semantics for first-degree entailments and ‘coupled trees’. *Philosophical Studies*, 29:149–168, 1976. Reprinted in Reprinted in *New Essays on Belnap-Dunn Logic*, Springer, 2019.

[11] A. Loparic and N. da Costa. Paraconsistency, paracompleteness and valuations. *Logique et Analyse*, 106:119–131, 1984.

[12] H. Omori and H. Wansing. 40 years of FDE: An introductory overview. *Studia Logica*, 105:1021–1049, 2017.

[13] A. Rodrigues and H. Antunes. First-order logics of evidence and truth with constant and variable domains. doi: 10.1007/s11787-022-00306-8 *Logica Universalis*, 2022.

[14] A. Rodrigues, J. Bueno-Soler, and W. Carnielli. Measuring evidence: a probabilistic approach to an extension of Belnap-Dunn Logic. *Synthese*, 198:5451–5480, 2020.

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TILING PROBLEMS AND COMPLEXITY OF LOGICS

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1. Introduction

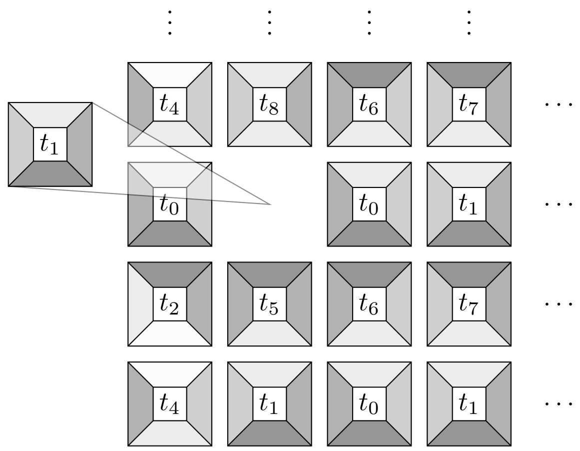
Domino, or tiling, problems [1, 9] provide us with a rich tool allowing to estimate bounds for computational complexity of problems arising in different fields of mathematics, in particular, in algebra [4, 10] and mathematical logic [2, 8, 14, 11, 6]. Sometimes, properties of tilings of some kind can be quite easily expressed in a formal language, and their description can be more elegant than, say, of Turing machines (or other computational models). Indeed, to describe a tiling, we only have to say that, for every tile, there are appropriate tiles on the top and on the right, and that moving right-top or top-right we see the same tile, while for a Turing machine, to describe just a configuration on some step of computation, we have to describe a head position, a state, and symbols stored in tape cells.

Here, we consider two tiling problems, known to be, respectively, Π^0_1-complete and Σ^1_1-complete, and show examples of their simulation in first-order theories and logics whose langages are enriched with some extra expressive means [8] but restricted in the number of individual variables, the number of predicate letters, and their arity.

2. Tiling problems we consider

We may think of a tile as a colored 1×1 square, with a fixed orientation. Each edge is colored. A tile type t consists of a specification of a color for each edge; we write $\boxed{\triangleright}t$, $\boxed{\triangleleft}t$, $\boxed{\triangledown}t$, and $\boxed{\vartriangle}t$ for the colors of, respectively, the left, the right, the top, and the bottom edges of the tiles of type t.

Let $T = \{t_0, \dots, t_n\}$ be a set of tile types. Informally, a T-tiling is an arrangement of tiles, whose types are in T, on a grid so that the edge colors of the adjacent tiles match, both horizontally and vertically; see the picture below (tile-holders are in the left-bottom corners of tiles).



The fist tiling problem we consider is the following: given a set $T = \{t_0, \dots, t_n\}$ of tile types, we are to determine whether there exists a T-tiling $f\colon \mathbb{N} \times \mathbb{N} \to T$ such that, for every $i, j \in \mathbb{N}$,

(1) $\boxed{\triangleleft}f(i,j) = \boxed{\triangleright}f(i+1,j)$;
(2) $\boxed{\triangledown}f(i,j) = \boxed{\vartriangle}f(i,j+1)$.

This problem is Π^0_1-complete [1]. The second tiling problem we consider can be obtained from the first one by adding an extra requirement

(3) the set $\{j \in \mathbb{N} : f(0,j) = t_0\}$ is infinite,

i.e., claiming that there are infinitely many tiles of type t_0 in the leftmost column. This problem is Σ^1_1-complete [9].

3. Classical theories

Assume, for simplicity, a classical first-order language with an infinite supply of monadic predicate letters $P_0, P_1, P_2, \ldots$ and two binary predicate letters H and V. The intending meaning of $P_k(x)$ is "x is placed with a tile of type t_k"; also, $H(x, y)$ means "y is to the right of x", and $V(x, y)$ means "y is above x". To describe an $\mathbb{N} \times \mathbb{N}$ grid, it is sufficient to say

$$\forall x \exists y\, H(x, y), \quad \forall x \exists y\, V(x, y), \quad \forall x \forall y\, (\exists z\, (H(x, z) \wedge V(z, y)) \leftrightarrow \exists z\, (V(x, z) \wedge H(z, y)).$$

Then, we can say that we are given a T-tiling:

- Each tile-holder holds a unique tile: $\forall x \bigvee_{i=0}^{n} (P_i(x) \wedge \bigwedge_{j \neq i} \neg P_j(x)).$
- The condition (1) for a T-tiling is satisfied: $\forall x \bigwedge_{i=0}^{n} (P_i(x) \to \forall y\, (H(x, y) \to \bigvee_{\text{◨} t_i = \text{◧} t_j} P_j(y))).$
- The condition (2) for a T-tiling is satisfied: $\forall x \bigwedge_{i=0}^{n} (P_i(x) \to \forall y\, (V(x, y) \to \bigvee_{\text{⬒} t_i = \text{⬓} t_j} P_j(y))).$

It is not hard to see that the conjunction of the above formulas is satisfiable if, and only if, there exists a T-tiling $f \colon \mathbb{N} \times \mathbb{N} \to T$ satisfying conditions (1) and (2). As a result, the Church's theorem [3] for the classical first-order logic follows. Since we can simulate all the predicate letters with a single binary one without adding extra individual variables [15, 16], this gives us a short proof of the known refinement [23] of the Church's theorem: the satisfiability problem is undecidable for languages with a single binary predicate letter and three individual variables. Moreover, we readily obtain undecidability (Σ^0_1-hardness) for infinite classes of theories of a binary predicate, again, with three individual variables [15, 16].

Observe that, with the use of Compactness theorem, the existence of a T-tiling satisfying (1) and (2) is equivalent to the existance, for every $n \in \mathbb{N}$, of an $n \times n$ tiling with T-tiles satisfying (1) and (2) for all appropriate i and j. Therefore, we can use only finitely many tile-holders (but their number must be unbounded). This observation allows us to simulate T-tilings on finite models and, thus, to obtain the Trakhtenbrot's theorem [24, 25] for satisfiability over finite models. Again, modulo some linguistic machinations, we obtain undecidability (Π^0_1-harness) for large classes of theories of a binary predicate defined by infinite classes of finite models [15, 16].

Notice that undecidability of some the theories — both Σ^0_1-hardness and Π^0_1-hardness — follow also from proofs like in [5, 13] by means of a general technique described in [22].

4. Classical theories with extra non-elementary expressive means

Having enriched the language with equality and the operator of transitive closure, we can use the transitive closure V^+ of V allowing us to express (3):

$$\exists x \forall y\, (V^+(x, y) \to \exists z\, (z \neq y \wedge V^+(y, z) \wedge P_0(z))).$$

Notice that equality can be eliminated if we add the condition of irreflexivity, i.e., $\forall x\, \neg V(x, x)$; also, variable z can be replaced with x. Then, adding the operator of composition $\circ$ of binary relations, we are able to express that moving right-top and top-right, we see the same tile, using the formula $\forall x \forall y\, ([V \circ H](x, y) \leftrightarrow [H \circ V](x, y))$, which contains only two individual variables. Again, using additional techniques, we can prove that the satisfiability for languages with a single binary relation, equality, the operators of transitive closure and composition is Σ^1_1-hard even for formulas with two variables [15]. Sometimes, the operator of transitive closure can be replaced with the operator asserting the transitivity of a binary relation [16].

5. Some remarks and further results

Examples of the use of tiling problems for obtaining results on the algorithmic complexity of various logics, both propositional and predicate, can be found in [2, 14, 6, 11, 19, 20, 21, 17, 18]. In particular, the tiling problems considered here can be used to obtain complexity results for theories of trees [18] and to prove that modal predicate logics whose Kripke frames are Noetherian orders are Π^1_1-hard in rather poor languages [17]; the latter result gives us an alternate argument for Kripke incompleteness of the predicate counterpart of the Gödel–Löb logic **GL** [12].

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References

[1] R. Berger. *The Undecidability of the Domino Problem*, volume 66 of *Memoirs of AMS*. AMS, 1966.
[2] E. Börger, E. Grädel, Y. Gurevich. *The Classical Decision Problem*. Springer, 1997.
[3] A. Church. A note on the "Entscheidungsproblem". *The Journal of Symbolic Logic*, 1:40–41, 1936.
[4] J. Conway, J. Lagarias. Tiling with polyominoes and combinatorial group theory. *J. Combin. Theory Ser. A*, 53(2):183–208, 1990.
[5] Yu. L. Ershov, I. A. Lavrov, A. D. Taimanov, M. A. Taitslin. Elementary theories. *Russian Mathematical Surveys*, 20(4):35–105, 1965.
[6] D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. *Many-Dimensional Modal Logics*, volume 48 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2003.
[7] D. Gabbay, V. Shehtman, D. Skvortsov. *Quantification in Nonclassical Logic, Volume 1*, volume 153 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2009.
[8] R. Grädel, M. Otto, E. Rosen. Undecidability results on two-variable logics. *Archive for Mathematical Logic*, 38:313–354, 1999.
[9] D. Harel. Effective transformations on infinite trees, with applications to high undecidability, dominoes, and fairness. *Journal of the ACM*, 33(224–248), 1986.
[10] J. Kari, P. Papasoglu. Deterministic aperiodic tile sets. *Geometric and functional analysis*, 9:353–369, 1999.
[11] R. Kontchakov, A. Kurucz, M. Zakharyaschev. Undecidability of first-order intuitionistic and modal logics with two variables. *Bulletin of Symbolic Logic*, 11(3):428–438, 2005.
[12] F. Montagna. The predicate modal logic of provability. *Notre Dame Journal of Formal Logic*, 25(2):179–189, 1984.
[13] A. Nies. Undecidable fragments of elementary theories. *Algebra Universalis*, 35:8–33, 1996.
[14] M. Reynolds, M. Zakharyaschev. On the products of linear modal logics. *Journal of Logic and Computation*, 11(6):909–931, 2001.
[15] M. Rybakov. Computational complexity of binary predicate theories with a small number of variables in the language. *Doklady Mathematics*, 507(6):61–65, 2022.
[16] M. Rybakov. Binary predicate, transitive closure, two-three variables: shall we play dominoes? To appear in *Logical Investigations*. (In Russian)
[17] M. Rybakov. Predicate counterparts of modal logics of provability: High undecidability and Kripke incompleteness. To appear in *Logic Journal of the IGPL*.
[18] M. Rybakov. Trees as a tool for modelling undecidable problems. To appear in *Herald of Tver State University. Series: Applied Mathematics*. (In Russian)
[19] M. Rybakov, D. Shkatov. Algorithmic properties of first-order modal logics of the natural number line in restricted languages. In Nicola Olivetti, Rineke Verbrugge, Sara Negri, and Gabriel Sandu, editors, *Advances in Modal Logic*, volume 13. College Publications, 2020.
[20] M. Rybakov, D. Shkatov. Algorithmic properties of first-order modal logics of linear Kripke frames in restricted languages. *Journal of Logic and Computation*, 31(5):1266–1288, 2021.
[21] M. Rybakov, D. Shkatov. Undecidability of QLTL and QCTL with two variables and one monadic predicate letter. *Logical Investigations*, 27(2):93–120, 2021.
[22] S. Speranski. A note on hereditarily Π^0_1- and Σ^0_1-complete sets of sentences. *Journal of Logic and Computation*, 26(5):1729–1741, 2016.
[23] A. Tarski, S. Givant. *A Formalization of Set Theory without Variables*, volume 41 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, 1987.
[24] B. A. Trakhtenbrot. The impossibility of an algorithm for the decidability problem on finite classes. *Doklady AN SSSR*, 1950. (In Russian; English translation in [26])
[25] B. A. Trakhtenbrot. On recursive separability. *Doklady AN SSSR*, 88:953–956, 1953. (In Russian)
[26] B. A. Trakhtenbrot. Impossibility of an algorithm for the decision problem in finite classes. *American Mathematica Society Translations*, 23:1–5, 1963.

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ON ALGORITHMIC EXPRESSIVITY OF FINITE-VARIABLE FRAGMENTS OF INTUITIONISTIC MODAL LOGICS

MIKHAIL RYBAKOV AND DMITRY SHKATOV

1. Introduction

Modal and intuitionistic propositional logics are often poly-time embeddable into their own fragments with a few variables (typically, zero, one, or two), and similar embeddings are sometimes constructed of fragments of logics with special properties into finite-variable fragments of those logics. The literature on the topic is quite extensive [2, 27, 11, 12, 4, 29] and includes contributions by the authors of this paper [3, 14, 15, 16, 17, 18, 20, 19, 21, 22, 23, 24, 25].

As a result, the validity problem for such fragments is as computationally hard as the validity problem for the full logic. (In general, modal and superintuitionistic propositional logics, even linearly approximable ones, may have arbitrarily hard fragments with a few variables since, for every set $A \subseteq \mathbb{N}$, one can construct [26] a linearly approximable logic whose fragment with a few variables (typically zero, one, or two) recursively encodes A. We obtain here similar embeddings for the intuitionistic modal logics **FS** and **MIPC**, introduced by, respectively, Fisher Servi [7] and Prior [13]. These logics have been introduced as counterparts of bimodal propositional logics, and can also be viewed as fragments of the predicate intuitionistic logic **QInt** (for details, see [10]); we note that this is not the only approach to constructing modal intuitionistic logics, cf. [5, 6, 28]. The complexity of **FS** and **MIPC** remains unresolved, but the results presented here show that single-variable fragments of these logics have the same complexity as the full logics.

2. Preliminaries

The intuitionistic modal language contains a countable set $\mathcal{P}$ of propositional variables, the constant $\bot$, binary connectives $\wedge$, $\vee$, and $\to$, and unary modal connectives $\Diamond$ and $\Box$. Formulas are defined in the usual way. A formula is *positive* if it does not contain occurrences of $\bot$. The set of propositional variables of a formula φ is denoted by $\mathit{var}\,\varphi$. The result of substituting a formula ψ for a variable p into a formula φ is denoted by $[\psi/p]\varphi$. The modal depth of a formula φ, denoted by $\mathit{md}\,\varphi$, is the maximal number of nested modal connectives in φ. The length of a formula φ, defined as the number of symbols in φ (with the binary encoding of variables), is denoted by $|\varphi|$.

We define the logics **FS** and **MIPC** semantically. A *Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a non-empty set of *worlds* and R is a partial order on W. An **FS**-*frame* is a triple $\mathfrak{F} = \langle W, R, \delta\rangle$, where $\langle W, R\rangle$ is a Kripke frame and δ is a map associating with each $w \in W$ a structure $\langle \Delta_w, S_w\rangle$, with Δ_w being a non-empty set of *points* and S_w a binary relation on Δ_w such that, for every $w, v \in W$,

$$v \in R(w) \;\Rightarrow\; \Delta_w \subseteq \Delta_v \quad\text{and}\quad S_w \subseteq S_v$$

An **FS**-frame $\mathfrak{F} = \langle W, R, \delta\rangle$ is an **MIPC**-*frame* if $S_w = \Delta_w \times \Delta_w$, for every $w \in W$. A *valuation* on an **FS**-frame $\langle W, R, \delta\rangle$ is a map associating with each $w \in W$ and each $p \in \mathcal{P}$ a subset $V(w, p)$ of Δ_w in such a way that

$$v \in R(w) \;\Rightarrow\; V(w, p) \subseteq V(v, p).$$

The pair $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$, where $\mathfrak{F}$ is an **FS**-frame and V a valuation on $\mathfrak{F}$, is called an **FS**-*model*. An **MIPC**-*model* is an **FS**-model over an **MIPC**-frame. The *truth-relation* $\models$ is defined by recursion (here, $\mathfrak{M}$ is a model, $w \in W$, $x \in \Delta_w$, and φ is a formula):

- $\mathfrak{M}, w, x \models p \;\leftrightharpoons\; x \in V(w, p)$ if $p \in \mathcal{P}$;
- $\mathfrak{M}, w, x \not\models \bot$;
- $\mathfrak{M}, w, x \models \varphi_1 \wedge \varphi_2 \;\leftrightharpoons\; \mathfrak{M}, w, x \models \varphi_1$ and $\mathfrak{M}, w, x \models \varphi_2$;
- $\mathfrak{M}, w, x \models \varphi_1 \vee \varphi_2 \;\leftrightharpoons\; \mathfrak{M}, w, x \models \varphi_1$ or $\mathfrak{M}, w, x \models \varphi_2$;
- $\mathfrak{M}, w, x \models \varphi_1 \to \varphi_2 \;\leftrightharpoons\; \mathfrak{M}, v, x \not\models \varphi_1$ or $\mathfrak{M}, v, x \models \varphi_2$ whenever $v \in R(w)$;
- $\mathfrak{M}, w, x \models \Diamond\varphi_1 \;\leftrightharpoons\; \mathfrak{M}, w, y \models \varphi_1$, for some $y \in S_w(x)$;
- $\mathfrak{M}, w, x \models \Box\varphi_1 \;\leftrightharpoons\; \mathfrak{M}, v, y \models \varphi_1$ whenever $v \in R(w)$ and $y \in S_v(x)$.

A formula φ is *true* in a model $\mathfrak{M}$ (notation: $\mathfrak{M} \models \varphi$) if $\mathfrak{M}, w, x \models \varphi$, for every world w of $\mathfrak{M}$ and every point x of w. A formula φ is *valid* an **FS**-frame $\mathfrak{F}$ if φ is true in every model over $\mathfrak{F}$. Logics **FS** and **MIPC** are defined as sets of formulas valid on, respectively, every **FS**-frame and every **MIPC**-frame.

3. Main results

In this section, we prove that logics **FS** and **MIPC** are polynomial-time embeddable into their own fragments with a single propositional variable. We first poly-time embed these logics into their own positive fragments. Let φ be a formula and $f \in \mathcal{P} \setminus var\,\varphi$. Define

$$\varphi^f = [f/\bot]\varphi; \qquad F_1 = \Diamond^{\leqslant md\,\varphi} f \to f; \qquad F_2 = f \to \Box^{\leqslant md\,\varphi} f; \qquad F_3 = \bigwedge_{p \in var\,\varphi} \Box^{\leqslant md\,\varphi}(f \to p),$$

and put $F = F_1 \wedge F_2 \wedge F_3$.

Lemma 1. *Let φ be a formula, $f \in \mathcal{P} \setminus var\,\varphi$, and $L \in \{\mathbf{FS}, \mathbf{MIPC}\}$. Then,*

$$\varphi \in L \iff F \to \varphi^f \in L.$$

Since φ^f and F are both positive, the map $e\colon \varphi \mapsto (F \to \varphi^f)$ embeds **FS** and **MIPC** into their own positive fragments.

We next define a polytime computable function $\cdot^*$ from the set of positive formulas to the set of one-variable positive formulas and show that, for $L \in \{\mathbf{FS}, \mathbf{MIPC}\}$ and every positive φ,

$$\varphi^* \in L \iff \varphi \in L.$$

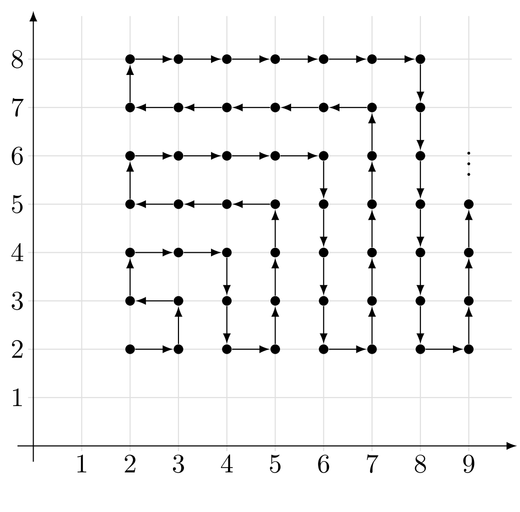
Hence, for every φ,

$$\varphi \in L \iff e(\varphi) \in L \iff e(\varphi)^* \in L.$$

The formula φ^* shall be obtain from φ using a substitution. We next define the formulas that shall be substituted for propositional variables of φ. These formulas, except G_1, G_2, and G_3, are divided into 'levels', indexed by elements of $\mathbb{N}$; formulas of level 0 are denoted A_i^0 or B_i^0, those of level 1, by A_i^1 and B_i^1, etc. We begin with G_1, G_2, and G_3, as well as formulas of levels 0 and 1:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
G_1 & = & \Diamond p; & A_1^1 & = & A_1^0 \wedge A_2^0 \to B_1^0 \vee B_2^0; \\
G_2 & = & \Diamond p \to p; & A_2^1 & = & A_1^0 \wedge B_1^0 \to A_2^0 \vee B_2^0; \\
G_3 & = & p \to \Box p; & A_3^1 & = & A_1^0 \wedge B_2^0 \to A_2^0 \vee B_1^0; \\
A_1^0 & = & G_2 \to G_1 \vee G_3; & B_1^1 & = & A_2^0 \wedge B_1^0 \to A_1^0 \vee B_2^0; \\
A_2^0 & = & G_3 \to G_1 \vee G_2; & B_2^1 & = & A_2^0 \wedge B_2^0 \to A_1^0 \vee B_1^0; \\
B_1^0 & = & G_1 \to G_2 \vee G_3; & B_3^1 & = & B_1^0 \wedge B_2^0 \to A_1^0 \vee A_2^0. \\
B_2^0 & = & A_1^0 \wedge A_2^0 \wedge B_1^0 \to G_1 \vee G_2 \vee G_3; & & &
\end{array}$$

We proceed by recursion. Let $k \geqslant 1$. Suppose the formulas $A_1^k, \dots, A_{n_k}^k$ and $B_1^k, \dots, B_{n_k}^k$ have been defined, with n_k being the number of formulas of the form A_i^k and, also, the number of formulas of the form B_i^k (e.g., if $k = 1$, then $n_k = 3$; the recursive definition for the cases where $k \geqslant 2$ is to be given). Define a linear order $\prec$ on the set $(\mathbb{N} \setminus \{0,1\}) \times (\mathbb{N} \setminus \{0,1\})$ as in the following picture, so that $\langle i, j\rangle \prec \langle i', j'\rangle$ if, and only if, there exists a path along one or more arrows from $\langle i, j\rangle$ to $\langle i', j'\rangle$:



We can then define an enumeration g of the pairs $\langle i, j\rangle \in (\mathbb{N} \setminus \{0,1\}) \times (\mathbb{N} \setminus \{0,1\})$ according to $\prec$, i.e., so that $g(2,2) = 1$, $g(3,2) = 2$, $g(3,3) = 3$, $g(2,3) = 4$, etc. Now, for every $i, j \in \{2, \dots, n_k\}$, define

$$A^{k+1}_{g(i,j)} \quad = \quad A^k_1 \to B^k_1 \vee A^k_i \vee B^k_j; \qquad B^{k+1}_{g(i,j)} \quad = \quad B^k_1 \to A^k_1 \vee A^k_i \vee B^k_j,$$

and let n_{k+1} be the number of the formulas of the form A^{k+1}_i (which is the same as the number of formulas of the form B^{k+1}_i) so defined; notice that $n_{k+1} = (n_k - 1)^2$. This completes the recursive definition of A^k_i and B^k_i.

Next, put

$$l_0 \quad = \quad |A^0_1| + |B^0_1| + |A^0_2| + |B^0_2|.$$

Lemma 2. *There exists $k_0 \in \mathbb{N}$ such that $n_k > l_0 \cdot 5^k$ whenever $k \geqslant k_0$.*

Now, let φ be a positive formula with $var\,\varphi = \{p_1, \dots, p_s\}$. Let k_φ be the least integer k such that $|\varphi| < l_0 \cdot 5^k$. By Lemma 2, $n_{k_\varphi + k_0} > l_0 \cdot 5^{k_\varphi + k_0}$; hence,

$$n_{k_\varphi + k_0} > l_0 \cdot 5^{k_\varphi + k_0} > 5^{k_0} \cdot |\varphi| > |\varphi| \geqslant s.$$

Lastly, define φ^* to be the result of substituting into φ, for every $r \in \{1, \dots, s\}$, the formula $A^{k_\varphi + k_0}_r \vee B^{k_\varphi + k_0}_r$ for the variable p_r (this substitution is well defined since $n_{k_\varphi + k_0} > s$).

We next show that φ^* is poly-time computable from φ.

Lemma 3. *For every $k \geqslant 0$ and every $i \in \{1, \dots, n_k\}$,*

$$|A^k_i| < l_0 \cdot 5^k \quad \text{and} \quad |B^k_i| < l_0 \cdot 5^k.$$

Lemma 4. *The formula φ^* is computable in time polynomial in $|\varphi|$.*

Proof. It suffices to show that $|\varphi^*|$ is polynomial in $|\varphi|$. Since k_φ is the least integer k such that $|\varphi| < l_0 \cdot 5^k$, surely $l_0 \cdot 5^{k_\varphi - 1} \leqslant |\varphi|$, and so

$$l_0 \cdot 5^{k_\varphi + k_0} \leqslant 5^{k_0 + 1}|\varphi|.$$

By Lemma 3, for every $i \in \{1, \dots, n_{k_\varphi + k_0}\}$,

$$|A^{k_\varphi + k_0}_i| < l_0 \cdot 5^{k_\varphi + k_0} \leqslant 5^{k_0 + 1}|\varphi| \quad \text{and} \quad |B^{k_\varphi + k_0}_i| < l_0 \cdot 5^{k_\varphi + k_0} \leqslant 5^{k_0 + 1}|\varphi|.$$

Hence, $|\varphi^*| < 2 \cdot 5^{k_0+1}|\varphi|^2$. $\square$

To obtain the desired result, it remains to show the following:

Lemma 5. *Let $L \in \{\mathbf{FS}, \mathbf{MIPC}\}$. Then, for every positive formula φ,*

$$\varphi \in L \quad \Longleftrightarrow \quad \varphi^* \in L.$$

From Lemmas 1, 4, and 5, we immediately obtain the following:

Theorem 6. *Let $L \in \{\mathbf{FS}, \mathbf{MIPC}\}$. Then, there exists a polynomial-time computable function embedding L into its own positive one-variable fragment.*

Corollary 7. *Let $L \in \{\mathbf{FS}, \mathbf{MIPC}\}$. Then, the positive one-variable fragment of L is polytime-equivalent to L.*

The results presented here are not immediately applicable to obtaining the computational complexity of finite-variable fragments of intuitionistic modal logics since the complexity of full logics remains unknown (we are only aware of decidability results [9, 31, 30, 1, 8] for modal intuitionistic logics).

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References

[1] Natasha Alechina and Dmitry Shkatov. A general method for proving decidability of intuitionistic modal logics. *Journal of Applied Logic*, 4(3):219–230, 2006.

[2] Patrick Blackburn and Edith Spaan. A Modal Perspective on the Computational Complexity of Attribute Value Grammar. *Journal of Logic, Language, and Information*, 2:129–169, 1993.

[3] Alexander Chagrov and Mikhail Rybakov. How many variables does one need to prove PSPACE-hardness of modal logics? In Philippe Balbiani, Nobu-Yuki Suzuki, Frank Wolter, and Michael Zakharyaschev, editors, *Advances in Modal Logic 4*, pages 71–82. King's College Publications, 2003.

[4] Stéphane Demri and Philippe Schnoebelen. The complexity of propositional linear temporal logics in simple cases. *Information and Computation*, 174, 84–103, 2002.

[5] Kosta Došen. Negative modal operators in intuitionistic logic. *Publications de l'Institut Mathématique* 35, 3–14, 1984.

[6] Kosta Došen. Negative modal operators in intuitionistic logic. Negation as a modal operator. Reports on Mathematical Logic 20, 15–27, 1986.

[7] Gisèle Fischer Servi. On modal logic with an intuitionistic base. *Studia Logica*, 36(3):141–149, 1977.

[8] Marianna Girlando, Roman Kuznets, Sonia Marin, Marianela Morales, Lutz Straßburger. Intuitionistic S4 is decidable. arXiv preprint `arXiv:2304.12094`.

[9] Carsten Grefe. Fischer Servi's intuitionistic modal logic has the finite model property. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic*, volume 1, pages 85–98. CSLI Publications, 1998.

[10] Dov Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2003.

[11] Joseph Y. Halpern. The effect of bounding the number of primitive propositions and the depth of nesting on the complexity of modal logic. *Artificial Intelligence*, 75(2):361–372, 1995.

[12] Edith Hemaspaandra. The complexity of poor man's logic. *Journal of Logic and Computation*, 11(4):609–622, 2001.

[13] Arthur Prior. *Time and Modality*. Clarendon Press, Oxford, 1957.

[14] Mikhail Rybakov. Embedding of intuitionistic logic into its two-variable fragment and the complexity of this fragment. *Logical Investigations*, 11:247–261, 2004. (In Russian)

[15] Mikhail Rybakov. Complexity of intuitionistic and Visser's basic and formal logics in finitely many variables. In Guido Governatori, Ian M. Hodkinson, and Yde Venema, editors, *Advances in Modal Logic 6*, pages 393–411. College Publications, 2006.

[16] Mikhail Rybakov. Complexity of the constant fragment of the propositional dynamic logic. *Herald of Tver State University. Series: Applied Mathematics*, 5:5–17, 2007. (In Russian)

[17] Mikhail Rybakov. Complexity of finite-variable fragments of EXPTIME-complete logics. *Journal of Applied Non-classical logics*, 17(3):359–382, 2007.

[18] Mikhail Rybakov. Complexity of intuitionistic propositional logic and its fragments. *Journal of Applied Non-Classical Logics*, 18(2–3):267–292, 2008.

[19] Mikhail Rybakov and Dmitry Shkatov. Complexity and expressivity of branching- and alternating-time temporal logics with finitely many variables. In B. Fischer B. and T. Uustalu, editors, *Theoretical Aspects of Computing–ICTAC 2018*, volume 11187 of *Lecture Notes in Computer Science*, pages 396–414, 2018.

[20] Mikhail Rybakov and Dmitry Shkatov. Complexity and expressivity of propositional dynamic logics with finitely many variables. *Logic Journal of the IGPL*, 26(5):539–547, 2018.

[21] Mikhail Rybakov and Dmitry Shkatov. On complexity of propositional linear-time temporal logic with finitely many variables. In J. van Niekerk and B. Haskins, editors, *Proceedings of SAICSIT2018*, pages 313–316. ACM, 2018.

[22] Mikhail Rybakov and Dmitry Shkatov. Complexity of finite-variable fragments of propositional modal logics of symmetric frames. *Logic Journal of the IGPL*, 27(1):60–68, 2019.

[23] Mikhail Rybakov and Dmitry Shkatov. Complexity of finite-variable fragments of products with K. *Journal of Logic and Computation*, 31(2):426–443, 2021.

[24] Mikhail Rybakov and Dmitry Shkatov. Complexity of finite-variable fragments of products with non-transitive modal logics. *Journal of Logic and Computation*, 32(5):853–870, 2022.

[25] Mikhail Rybakov and Dmitry Shkatov. Complexity of finite-variable fragments of propositional temporal and modal logics of computation. *Theoretical Computer Science*, 925:45–60, 2022.

[26] Mikhail Rybakov and Dmitry Shkatov. Complexity function and complexity of validity of modal and superintuitionistic propositional logics. To appear in *Journal of Logic and Computation*, `https://doi.org/10.1093/logcom/exac085`.

[27] Edith Spaan. Complexity of Modal Logics. PhD thesis. University of Amsterdam, 1993.

[28] Stanislav O. Speranski. Negation as a modality in a quantified setting. *Journal of Logic and Computation* 31(5), 1330–1355, 2021.

[29] Vítěslav Švejdar. The decision problem of provability logic with only one atom. *Archive for Mathematical Logic*, 42(8):763–768, 2003.

[30] Frank Wolter and Michael Zakharyaschev. Intuitionistic modal logics as fragments of classical bimodal logics. In E. Orlowska, editor, *Logic at Work*, pages 168–186. Springer, Berlin, 1999. p

[31] Frank Wolter and Michael Zakharyaschev. Modal description logics: modalizing roles. *Fundamenta Informaticae*, 39(4):411–438, 1999.

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MULTI-AGENT MODAL LOGIC AND UNCERTAIN INFORMATION

VLADIMIR V. RYBAKOV

Abstract

This paper studies relational Kripke models for computation truth values of formulas in some extended logical language. The multi-agent modal language is extended by introduction a new modal operation $D(\alpha, \beta)$. The truth of this operation is determined in a way by implicit information: it is true only if the number of states where the formula α is true is strictly less than the number of states where the formula β is true. From mathematical point of view, the paper investigates satisfiability problem for formulas in such special language, we construct a mathematical algorithm verifying their satisfiability.

Keywords: *knowledge, information, modal logics, relational Kripke models, satisfiability problem*

General comment

Computer science and Information Sciences use technique of mathematical symbolic logic; e.g. logic is an effective tool in research and development of software, analysis of correctness the information. Several special domains are very useful here, for example it is the temporal logic (cf. Abadi M., Manna Z. Temporal logic programming, multi-modal logics, multi-agents' logics (cf. Ramy Shahin, Based Event Languages for the Development of Reactive Application Systems, – Towards Modal Software Engineering). For such applications some special variations of non-classical logic have been developed, e.g. it is the linear temporal logic LTL, which got to be very popular in computer science immediately after its introduction. Multi-agents' logics are often used in areas of analysis various obtained information, on its reliability and safety.

Results

To briefly define used notation and definition, to make extended abstract easy readable; we start from semantic models. We first fix an interval partitioning In of the set of all natural numbers N, so In is a set of indexes and

$$N = \bigcup_{i \in In} [c_i, c_{i_1}], c_i \ < \ c_{i+1}.$$

That will be our base set — worlds, states.

The valuations V of a set $Prop$ of propositional variables (letters) in such frames $\mathcal{F}^{FP}$ defined as follows,

$$\forall p \in Prop,\ V(p) \subseteq \bigcup_i [c_i, c_{i_1}].$$

The starting language of our logical system is the standard modal language of multi-agents' modal logic, that is it contains a finite number of modalities like $\Diamond_j$ for any agent j. Any $\Diamond_j$ refers to some linear accessibility relation which each is a part of standard linear order on segments $[c_i, c_{i_1}]$ and generally speaking is its own for each j.

Besides our approach uses introduction of a new modal binary operation $D(\alpha, \beta)$ with the following meaning and definition:

$$(\mathcal{M}^{FP}, x) \models_{V_j} D(\alpha, \beta) \iff \exists i \in N : x \in [c_i, c_{i+1}]$$

and the number of states on the segment $[c_i, c_{i+1}]$ in which the formula α is true w.r.t. valuation V_j, is strictly less then the number of states of this segment at which the formula β is true.

The binary logical operation $D(\alpha, \beta)$ is used in an unusual way. We may interpret it as an expert assessments; that is a sort of comparison statements in an implicit situation – when precise amount of states where the statements are true is not known in precise numerical value in unknown.

Theorem 1. *A formula F is satisfiable in a model introduced earlier (cf. $M^{FP} = \langle \bigcup_{i \in In} [c_i, c_{i_1}] \rangle$) iff it is satisfiable in a special model based at a frame of kind $\mathcal{F}^{FP}$ in its interval $[c_1, c_2]$ size of which is computable from the size of F.*

Using this theorem, as a consequence we obtain

Theorem 2. *The problem of satisfiability for formulas $D(x, y)$ in the logic L^{FP} is decidable.*

It is important to say here that we do not claim such results for a kind of whole logic which would include arbitrary formulas of kind $D(\alpha, \beta)$. We consider here, in the case, only formulas α and β which themselves already do not include formulas of sort $D(x, y)$. That is a special precaution in order to avoid infinite loop in proofs which was made in the very beginning of this research. We compare pure modal formulas with unknown amount of satisfying them states.

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SOME REMARKS ON WELL-STRUCTURED TRANSITION SYSTEMS

NIKOLAY V. SHILOV

1. Well-pre-ordered Transition Systems

Let D be a set. A *pre-order* (or *quasi-order*) is a reflexive and transitive binary relation on D. A *well-pre-order* (or *well-quasi-order*) is a pre-order $\preceq$ such that every infinite sequence $d_0, \dots d_i, \dots$ in D contains a pair of elements d_m and d_n so that $m < n$ and $d_m \preceq d_n$.

For sake of completeness, let us extend some standard concepts from ordered sets onto pre-ordered sets. Let $(D, \preceq)$ be a well-pre-ordered set. An *ideal* (or *upward-cone* or *cone*) is an upward closed subset of D, i.e., any set $I \subseteq D$ such that for all $d', d'' \in D$, if $d' \preceq d''$ and $d' \in I$ then $d'' \in I$. Every $d \in I$ generates the upward cone $(\uparrow d) \equiv \{e \in D \ : \ d \preceq e\}$. For every set $S \subseteq D$ any element $d \in S$ is a *minimal element* of S, if for every element $s \in S$ either $d \preceq s$ or d and s are non-comparable. For any subset $S \subseteq D$, the set of its minimal elements is $min(S)$; a *basis* of S is any subset $B \subseteq S$ such that for every $s \in S$ there exists an element $b \in B$ such that $b \preceq s$.

Let us present some algebraic easy to prove properties of well-pre-orders [1, 7]. Let us fix any well-pre-ordered set $(D, preceq)$. Firstly, $(D, \preceq)$ is well-founded, i.e., any infinite strictly decreasing sequence of elements of D is impossible; moreover, every infinite sequence in $(D, \preceq)$ contains an infinite non-decreasing subsequence of comparable elements. Next, every subset $S \subseteq D$ provided with the pre-order $\preceq$ is well-pre-ordered set also. Third, every $S \subseteq D$ has a finite basis that comprises all minimal elements $min(S)$; in particular, every ideal I has a finite basis $min(I)$, and $I = \bigcup_{d \in min(I)} (\uparrow d)$. Finally, every non-decreasing sequence of ideals $I_0 \subseteq \dots \subseteq I_i \subseteq \dots$ eventually stabilizes, i.e., there is some $k \geq 0$ such that $I_m = I_n$ for all $m, n \geq k$.

Let *Act* be any fixed finite alphabet of action symbols. A *transition system* (or *Kripke frame*) is a tuple (D, R), where the domain D is any non-empty set of elements (that are called *states* or *worlds*), and the *interpretation* R is a total mapping $R : Act \to 2^{D \times D}$. A run (in the frame) is a maximal sequence of states $s_0 \dots s_i s_{i+1} \dots$ such that for all adjacent states within the sequence $(s_i, s_{i+1}) \in R(a)$ for some $a \in Act$. A well-pre-ordered transition system (WPTS) is a triple $(D, \preceq, R)$ such that $(D, \preceq)$ is a well-pre-ordered set and (D, R) is a Kripke frame.

There are 4 options for *compatibility* of the well-pre-order $\preceq$ and the interpretation $R(a)$ of an action symbol $a \in Act$, they are depicted in Fig. 4 in logic, algebraic, and diagram notation. The terminology introduced in this figure can be explained as follows. The adjectives *upward* and *downward* have been introduced in [7], they are applicable to $\preceq$ and have an explicit and evident mnemonics. The adjective *future* is about states *after* action(s), i.e., future states, while states *before* an action are the past states.

In [9] the compatibility conditions are called *Fisher Servi conditions* because of intuitionistic modal logic suggested by G. Fisher Servi [8] (see also [14]): semantics of Fisher Servi logic is defined in partially ordered transition systems $(D, \preceq, R)$, where $\preceq$ is a partial order which is upward and downward compatible with R.

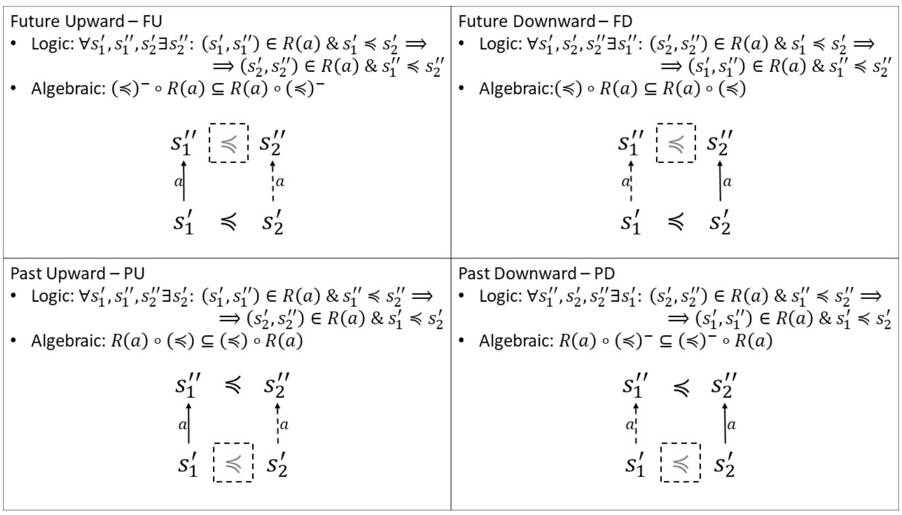

Figure 4. Pre-order-and-transition compatibility

But recently we learned from [6] that very similar compatibility conditions were introduced and studied by the same time by other researchers (ex., [5]). Also, in recent papers on well-structured transition system (ex., [3]) the conditions and their variations are referred as *monotony conditions.* Because of it, let us refer the conditions presented in the Fig. 4.

An upward compatible well-pre-ordered transition system is called *Well-Structured Transition System* (WSTS). Extensive case study and some generic examples of WSTS with a single action symbol can be found in the foundational papers [1, 7]. Variations of WSTS is a topic for further studies in the context of formal verification: for example, paper [3] presents results based on assumption that monotony should be applied to states that are reachable one from another.

Simulation and bisimulation [12] can be defined in terms of compatibility conditions [9].

- Future upward compatibility with a binary relation $\preceq$ means that the relation is a *simulation* relation on the states of the transition system (D, R).
- Future downward compatibility with the *inverse* of a binary relation $\preceq$ means that the *inverse relation* $(\preceq)^-$ is a simulation relation on the states of the transition system (D, R).
- Future upward compatibility with an *equivalence relation* $\simeq$ means that the relation $\simeq$ is a *bisimulation* on the states of the transition system (D, R).

2. Model Checking in Well-Structured Transition Systems

Model checking is an algorithmic problem to find (to compute) semantics of a given formula (of some given logic) in a given model (for this logic). In [9] the following *Theorem* had been proven.

Theorem 1. *The model checking problem is decidable for disjunctive formulas of the Propositional μ-Calculus in intuitionistic models over well-structured transition systems with decidable well-pre-order and tractable past.*

The original formulation in [9] differs from the presented here, but actually is equivalent. Let us define below concepts that are used in the Theorem statement.

Let us start with definitions for *decidable well-pre-order* and *tractable past.* The *decidability for the well-pre-order* means that $\preceq$ is decidable as a binary relation on states. *Tractable past* [9] means that for any $a \in Act$ the function $\lambda s \in D\ :\ \min\{t \in D\ :\ (t, s) \in R(a)\}$ is computable.

Then let us define *intuitionistic models.* Let Prp be an alphabet of *propositional variables.* A *(Kripke) model* (or *labeled transition system*) over a frame $F = (D, R)$ (any well-pre-ordered or well-structured transition system $(D, \preceq, R)$ in particular) is F together with *interpretation* (or *valuation*) $I : Prp \to 2^D$ of individual propositional variables by sets of states. Interpretation is said to be *intuitianistic*, if the frame is a well-pre-ordered transition system and $I(p)$ is an upward-cone (i.e., an ideal) for every propositional variable $p \in Prp$. A model (labeled transition system) is said to be *intuitianistic*, if has an intuitianistic interpretation.

The propositional μ-Calculus of D. Kozen (μC) [10, 11] is a very powerful propositional program logic with fix-points. It is widely used for specification and verification of properties of finite state systems.

The syntax of μC consists of formulae, a context-free definition of formulae follows:

$$\phi\ ::=\ p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid ([a]\phi) \mid (\langle a\rangle\phi) \mid (\nu p.\ \phi) \mid (\mu p.\ \phi)$$

where metavariables ϕ, p, and a range over formulae, propositional variables in Prp, and action symbols in Act respectively. The only context constraint is the following: no instances of a bound (by μ or ν) propositional variables are in the range of odd number of negations.

Disjunctive formulae of μC don't use negation, conjunction,box-modality, the construct of the greatest fix-point and, a context-free definition of disjunctive formulae follows: $\phi\ ::=\ p \mid (\phi \vee \phi) \mid (\langle a\rangle\phi) \mid (\mu p.\ \phi)$.

The semantics of μC is defined in labeled transition systems. In every model $M = (D, R, I)$ where (D, R) is a Kripke frame, and I is an interpretation, the semantics $M(\phi)$ of any formula ϕ is a subset of the domain D that is defined by induction on the formula structure as follows.

- $M(p) = I(p)$, $M(\neg\phi) = D \setminus M(\phi)$,
  $M(\psi \wedge \theta) = M(\psi) \cap M(\theta)$, and $M(\psi \vee \theta) = M(\psi) \cup M(\theta)$
- $M([a]\psi) = \{s\ :\ \text{for every } t \in D, \text{ if}(s, t) \in R(a) \text{ then } t \in M(\psi)\}$,
  and $M(\langle a\rangle\psi) = \{s\ :\ \text{for some } t \in D\ (s, t) \in R(a) \text{ and } t \in M(\psi)\}$
- $M(\nu p.\ \psi) =$ the greatest fix-point of the mapping $\lambda S \subseteq D\ .\ \Big(M_{S/p}(\psi)\Big)$,
  and $M(\mu p.\ \psi) =$ the least fix-point of the mapping $\lambda S \subseteq D\ .\ \Big(M_{S/p}(\psi)\Big)$,
  where $M_{S/p}(\psi)$ is the model that agrees with M everywhere but p, $M_{S/p}(p) = I_{S/p}(p) = S$

3. Relations to Metacompilation and Speculative Computations

Metacompilation [13] is an approach to program analysis designed to detect some cases of program (infinite) looping basing on either bisimulation (i.e., models with well-structured transition systems) or *generalization* and *speculative computations* (to be explained later).

In this section we assume that the set of states D and interpretations R and I for action symbols Act and propositional variables Prp are fixed; because of this convention, let us write in this section

- $s \xrightarrow{a} t$ instead $(s,t) \in R(a)$,
- and $p(s)$ instead $s \in I(p)$.

Let us define a simple programming language. A program α is a finite set of labeled operators of two kinds:

- labeled assignment operator $l:\ a\ goto\ J$ where $l \in \mathbb{N}$ is a natural number (label marking/labeling this operator), $a \in Act$ is an action (the body/action of the operator), and $J \subseteq \mathbb{N}$ is a finite set of natural numbers (called jumps of the operator, may be the empty set)
- labeled conditional (choice) operator $l:\ if\ p\ then\ J\ else\ K$ where $l \in \mathbb{N}$ is a natural number (label marking/labeling this operator), $p \in Prp$ is a condition (the condition/test of the operator), and $J, K \subseteq \mathbb{N}$ are finite sets of natural numbers (called true/positive-jumps and false/negative-jumps of the operator, may be the empty set any or both).

The initial label (of α) is the least label that marks any labeled operator in the program; an exit label (of α) is any label that has instance(s) in α but does not mark any labeled operator in α.

To define semantics for programs, let us fix a program α.

A *configuration* (of the program α) is any pair (l, s) where l is a label (i.e., a natural number) and s is a state.

- A *firing of a labeled assignment operator* $(l : a\ goto\ J) \in \alpha$ is any pair of configurations $(l, s)(j, t)$ such that $s \xrightarrow{a} t$ and $j \in J$.
- A *positive firing of a labeled choice operator* $(l:\ if\ p\ then\ J\ else\ K) \in \alpha$ is any pair of configurations $(l, s)(j, s)$ such that $p(s)$ and $j \in J$.
- A *negative firing of a labeled choice operator* $(l:\ if\ p\ then\ J\ else\ K) \in \alpha$ is any pair of configurations $(l, s)(k, s)$ such that not $p(s)$ but $k \in K$.
- A *firing of a labeled conditional (choice) operator* $(l:\ if\ p\ then\ J\ else\ K) \in \alpha$ is any it's positive or negative firing.

A firing (of α) is any firing of any operator of the program.

A *run* (of α) is any (finite or infinite) sequence of configurations $(l_0, s_0)\dots(l_n, s_n)(l_{n+1}, s_{n+1})\dots$ such that for any consecutive pair of configurations $(l_n, s_n)(l_{n+1}, s_{n+1})$ within this sequence is a firing of the program α; configuration (l_0, s_0) is called starting configuration of the run, label l_0 – its starting label, and state s_0 – its starting state; if the run is finite and configuration (l_{n+1}, s_{n+1}) is the last in the run, then (l_{n+1}, s_{n+1}) is called the ending configuration, label l_{n+1} – its ending label, and state s_{n+1} – its ending state.

- An *initial run* is any run starting in the initial label.
- A *terminal run* is any finite run ending in any exit label.
- A *looping run* (or *diverging run*) is any infinite run.
- A *complete run* is either a terminal or a looping run.
- *Execution* is any initial complete run.
- An *executional run* is any run that is a sub-run (i.e., a sub-word) of some execution.

Let us extend the definition of bisimulation from frames to models as follows: *full bisimulation* is any bisimulation $\simeq$ such that for any two states if $s \simeq t$ then $p(s) \Leftrightarrow p(t)$ for every propositional variable.

Lemma 2. *For any program α, any label l, states s, t, and any full bisimulation $\simeq$, if $s \simeq t$ and α has a diverging run from (l, s) than it has a diverging run from (l, t).*

Generalization is any binary relation $\overset{gen}{\Longrightarrow}$ on D (i.e., $\overset{gen}{\Rightarrow} \subseteq D \times D$). – Remark that we use *asymmetric notation* for generalization because we do not assume the relation being symmetric (or being reflexive, transitive, antisymmetric, etc.). If $\overset{gen}{\Longrightarrow}$ is a generalization, then for any labels l, k, any states s, t let us write $(l, s) \overset{gen}{\Longrightarrow} (k, t)$ if $l = k$ and $s \overset{gen}{\Longrightarrow} t$. Let us say that a generalization $\overset{gen}{\Longrightarrow}$ is *sound* (for our fixed program α) if for any states $s, t, u \in D$ and any labels l, k, $s \overset{gen}{\Longrightarrow} t$ and firing $(l, t)(k, u)$ together imply that there exists a state $v \in D$ such that $v \overset{gen}{\Longrightarrow} u$ and $(l, s)(k, v)$ is a firing also; in this definition the firing $(l, t)(k, u)$ is called *generalized* or *speculative*, while the firing $(l, s)(k, v)$ is called an *actual firing*; in case of sound generalization any speculative firing always should be *supported* by some actual one. – It is easy to see that soundness is future downward compatibility from Fig. 4.

Lemma 3. *For any program α, any sound generalization $\overset{gen}{\Longrightarrow}$, any label l, and any actual state $s \in D$, and a generalized state $t \in D$ such that $s \overset{gen}{\Longrightarrow} t$, if there exists a finite speculative run ρ starting and ending in (l, t), then α has an infinite actual run that is generalizable to the infinite speculative run ρ^ω (i.e., the infinite iteration of ρ).*

The following method is based on the above Lemma 3 is known as *metacompilation*.

Precondition: α is a program, (k, r) is its configuration, and *Gen* is a set of sound generalizations.

Procedure: Let us construct all runs of α starting in (k, r) in depth-first manner. As soon as we detected in any path from (k, r) any two configurations (l, s) and (l, v) on a run that both can be generalized (using any generalization $\overset{gen}{\Longrightarrow}$ in *Gen*) to some speculative configuration (l, t), $(l, s) \overset{gen}{\Longrightarrow} (l, t)$ and $(l, v) \overset{gen}{\Longrightarrow} (l, t)$, such that there exists a speculative run starting and ending in this speculative configuration (l, t), then we may mark both configurations (l, s) and (l, v) in by a special label *loop*.

Invariant: For any configuration (l, s) reachable from (k, r), if the configuration is marked by *loop*, then there exists a looping run starting in (l, s).

Roughly speaking, metacopilation is attempting something á la cross-world predication [4]: it tries to run speculative computations but then make conclusions about actual ones.

References

[1] P. A. Abdulla, K. Cerans, B. Jonsson, T. Yih-Kuen. General decidability theorems for infinite-state systems. *In: Proc. 11th IEEE Symp. Logic in Computer Science (LICS'96). 313–321, 1996.*

[2] A. Arnold, D. Niwinski. Rudiments of μ-calculus. North Holland, 2001.

[3] B. Bollig, A. Finkel, A. Suresh. Branch-Well-Structured Transition Systems and Extensions. *In: Formal Techniques for Distributed Objects, Components, and Systems. FORTE 2022. Mousavi, M.R., Philippou, A. (eds). Lecture Notes in Computer Science, 13273. Springer, Cham. 50–66, 2022.*

[4] E. V. Borisov. A logic for cross-world predication. *In: Proc. of the Second Int. Congress of Russian Society of History and Philosophy of Science "Science as a public good", vol. 4, 205–209, 2020.*

[5] M. Božić, K. Došen. Models for normal intuitionistic modal logics. *Studia Logica, 43(3), 217–245, 1984.*

[6] S. Drobyshevich, S. Odintsov, H. Wansing. Moisil's modal logic and related systems. *In: Int. Conference "Maltsev Readings". September 20-24, 2021, Abstracts of Talks, 62–62, 2021.*

[7] A. Finkel, Ph. Schnoebelen. Well-structured transition systems everywhere! *Theoretical Computer Science, 256(1-2), 63–92, 2001.*

[8] G. Fischer Servi. Axiomatizations for Some Intuitionistic Modal Logics. *Rend. Sem. Mat. Univers. Polit., 42, 179–194, 1984.*

[9] E. V. Kouzmin, N. V. Shilov, V. A. Sokolov. Model checking *mu*-calculus in well-structured transition systems. *In: Proceedings. 11th International Symposium on Temporal Representation and Reasoning, 2004. TIME 2004., Tatihou, France. 152–155, 2004.* (Extended version — in *Bulletine of Novosibirsk Computing Center. Computer Science. 20, 49–59, 2004.*)

[10] D. Kozen. Results on the Propositional Mu-Calculus. *Theoretical Computer Science, 27(3), 333–354, 1983.*

[11] N.Ṽ. Shilov, K. Yi. How to find a coin: propositional program logics made easy. *In: Current Trends in Theoretical Computer Science, 2, 181–213, 2004.* (Preliinary version — in *The Bulletin of the European Association for Theoretical Computer Science, 75, 127–151, 2021.*)

[12] C. Stirling. The joys of bisimulation. *In: Mathematical Foundations of Computer Science 1998. MFCS 1998. Brim, L., Gruska, J., Zlatuška, J. (eds), Lecture Notes in Computer Science, Springer Verlag, 1450, 142–151, 1998.*

[13] V. F. Turchin. The concept of a supercompiler. *ACM Trans. Program. Lang. Syst. 8, 3, 292–325, 1986.*

[14] F. Wolter, M. Zakharyaschev. Intuitionistic Modal Logic. In: *Logic and Foundations of Mathematics. Cantini, A., Casari, E., Minari, P. (eds). Syntheses Library, vol 280. Springer, 1999.*

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NONASSOCIATIVE INTUITIONISTIC MULTIPLICATIVE EXPONENTIAL LINEAR LOGIC AND ITS EXTENSION

YIHENG WANG, XUE GE, AND ZHE LIN

Linear logic was introduced by Jean-Yves Girard [1] in 1987. Due to the constructive property and strong representation ability, linear logic is closely connected with theoretical computer science and linguistics, such as logic programming and natural language processing [2, 3, 4, 5, 6, 7]. The decision problems of linear logic fragments have been studied by many researchers for years [8, 9, 10]. As one of the most important fragments in linear logic, the decidability problem of multiplicative exponential linear logic (MELL) is a long-time open question. What makes MELL matter is the relationship between it and the reachability problem in computer science. It is known that the reachability problem for Petri nets [11], which can be encoded into MELL [12], is equivalent to the reachability problem of vector addition systems with states (VASS) [13]. The branching VASS (BVASS) [14] is a model of counter machines which matches MELL. In terms of the decidability problem of MELL, Ranko Lazić and Sylvain Schmitz [15] proved that the decidability problem of MELL is equivalent to the reachability problem for BVASS and its complexity is at least TOWER-hard. A.P. Kopylov [10] proved that the MELL with weakening rule i.e., affine logic, is decidable. Lutz Straßburger [16] showed that a conservative extension of MELL plus mix, plus nullary mix by a self-dual noncommutative connective called NEL is undecidable. Recently, Katalin Bimbó [17] gave the sequent calculus system of propositional MELL and its extension: contraction-free propositional MELL and claimed that propositional MELL is decidable. However, Lutz Straßburger [18] pointed out that there are mistakes in Bimbó's proof. He showed that the MELL with contraction is decidable in the same paper. Yet, the reachability problem of BVASS, or equivalently the decidability problem of MELL still remains open.

We continue this line of research. The main motivation of this paper comes from hints from some existing results related to the decidability problem of MELL. Taking a brief look at the history, one has that Patrick Lincoln et al. [8] proved propositional linear logic (LL) and its intuitionistic version (ILL) are undecidable. Recently, Hiromi Tanaka [19] showed the undecidability of the nonassociative noncommutative intuitionistic linear logic (NACILL) and its extensions with exchange and contraction rules. By encoding the finitary consequence relation in full nonassociative Lambek calculus (FNL) into the provability in NACILL, Tanaka showed that NACILL is a strong conservative extension of FNL. The nonassociative noncommutative linear logic (NACLL) is undecidable as well. One can see that there seems to be a corresponding relationship among these results. LL and ILL are undecidable, while their nonassociative and noncommutative versions are undecidable too. Further, it is clear that MELL and IMELL are the additive-free versions of LL and ILL, while nonassociative intuitionistic MELL (NIMELL) is the additive-free version of NACILL. Since LL and NACLL together with ILL and NACILL are all undecidable, it may indicate that there is also a relationship between NIMELL and IMELL. Therefore, research about the decidability problem of NIMELL is quite natural if we want to know the answer to the MELL decidability open question.

In this paper, We emphasize the decision problems of the fragments of the nonassociative propositional linear logic. By showing the cut elimination result for NIMELL with assumption Φ, we can prove the consequence relation of NIMELL is decidable in polynomial time. Further, we prove the interpolation lemma for NIMELL and its distributive lattice extension NDILL. We show that NDILL is decidable as well. Further up to the knowledge of the authors, the cut-free sequent calculus, interpolation lemma, and decidability problem of NIMELL are not well investigated. This paper could be regarded as a beneficial supplement and clue to the current research about the MELL decidability problem. Our method is inspired by [20, 21].

Definition 1. *The set of formulas $\mathcal{F}$ and set of formula structures $\mathcal{FS}$ are defined inductively as follows :*

$$\mathcal{F} \ni \alpha ::= p \mid !\alpha \mid \alpha \otimes \beta \mid \beta \leftarrow \alpha \mid \alpha \rightarrow \beta$$

*where $p \in$ **Var***

$$\mathcal{FS} \ni \Gamma ::= \alpha \mid \Gamma, \Delta$$

A sequent is an expression of the form $\Gamma \Rightarrow \alpha$ where Γ is a formula structure and α is a formula. A context is a formula structure $\Gamma[-]$ with a designated position $[-]$ which can be filled with a formula structure. In particular, a single position $[-]$ is a context. Let $\Gamma[\alpha]$ be a formula structure obtained from $\Gamma[-]$ by substituting formula α for $[-]$. By $!\Gamma$, we denote a new formula structure by replacing every formula α in Γ with $!\alpha$.

Definition 2. *The Gentzen-style sequent calculus NIMELL consists of the following axiom and rules:*

(1) *Axiom:*

$$(\text{Id})\ \alpha \Rightarrow \alpha$$

(2) *Logical rules:*

$$\frac{\Delta \Rightarrow \alpha; \quad \Gamma[\beta] \Rightarrow \gamma}{\Gamma[\Delta, (\alpha \to \beta)] \Rightarrow \gamma}(\to \mathrm{L}) \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}(\to \mathrm{R})$$

$$\frac{\Gamma[\alpha] \Rightarrow \gamma; \quad \Delta \Rightarrow \beta}{\Gamma[(\beta \leftarrow \alpha), \Delta] \Rightarrow \gamma}(\leftarrow \mathrm{L}) \quad \frac{\Gamma, \beta \Rightarrow \alpha}{\Gamma \Rightarrow \beta \leftarrow \alpha}(\leftarrow \mathrm{R})$$

$$\frac{\Gamma[\alpha, \beta] \Rightarrow \gamma}{\Gamma[\alpha \otimes \beta] \Rightarrow \gamma}(\otimes \mathrm{L}) \quad \frac{\Gamma \Rightarrow \alpha; \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \otimes \beta}(\otimes \mathrm{R})$$

$$\frac{\Gamma[\beta] \Rightarrow \alpha}{\Gamma[!\beta] \Rightarrow \alpha}(!\mathrm{L}) \quad \frac{!\Gamma \Rightarrow \alpha}{!\Gamma \Rightarrow !\alpha}(!\mathrm{R})$$

$$\frac{\Gamma[!\Delta, !\Delta] \Rightarrow \beta}{\Gamma[!\Delta] \Rightarrow \beta}(\mathrm{Con}_!) \quad \frac{\Gamma[\Delta] \Rightarrow \beta}{\Gamma[\Delta, !\Sigma] \Rightarrow \beta}(\mathrm{Weak}_!)$$

(3) *Cut rule:*

$$\frac{\Delta \Rightarrow \alpha; \quad \Gamma[\alpha] \Rightarrow \beta}{\Gamma[\Delta] \Rightarrow \beta}(\mathrm{Cut})$$

A sequent $\Gamma \Rightarrow \alpha$ is provable in NIMELL, denoted by $\vdash_{NIMELL} \Gamma \Rightarrow \alpha$, if there is a derivation of $\Gamma \Rightarrow \alpha$ in NIMELL.

Remark 3. *The exponential ! here only allows contraction and weakening rule, which means it is a subexponential rather than a full-power exponential, such as the exponential in Tanaka's NACILL* [19].

We assume the non-logical assumption is of the form $\alpha \Rightarrow \beta$. For a set Φ of sequents $\alpha \Rightarrow \beta$, NIMELL(Φ) denotes the system of NIMELL with all formulas from Φ as assumptions. By Φ-restricted cut rule, we mean the following rule:

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1[\beta] \Rightarrow \gamma}{\Gamma_1[\Gamma_2] \Rightarrow \gamma}(\Phi\mathrm{-Cut})$$

where $\alpha \Rightarrow \beta$ is an assumption in Φ.

We define a related sequent calculus of NIMELL(Φ), denoted by NIMELL$(\Phi)^r$: axiom and rules of inference are simply the rules of NIMELL(Φ) together with the Φ-restricted cut rule. By showing the cut elimination result for NIMELL$(\Phi)^r$, one has the extended subformula property (cf. [22]).

Theorem 4 (Cut Elimination)**.** *Every sequent which is provable in NIMELL$(\Phi)^r$ can be proved also in NIMELL$(\Phi)^r$ without any applications of (Cut) rule.*

Corollary 5 (Extended Subformula Property)**.** *For any provable sequent $\Gamma \Rightarrow \alpha$ in NIMELL(Φ), there exists a proof of $\Gamma \Rightarrow \alpha$ such that all formulas appearing in the proof are subformulas of formulas in Φ or $\Gamma \Rightarrow \alpha$.*

Let T be a finite set of formulas closed under subformulas which contains all formulas appearing in Φ. A T-sequent $\Gamma \Rightarrow \alpha$ is a sequent such that all formulas appearing in Γ and α belong to T. By the above results, one gets a decision procedure for the consequence relation of NIMELL and shows that it is decidable in polynomial time.

Definition 6. *Assume Φ is finite. Let $T_! = \{!\alpha \mid \alpha \in T\}$, $T^! = T \cup T_!$. A sequent is said to be basic if it is a $T^!$-sequent of the form $\alpha, \beta \Rightarrow \gamma$ or $\alpha \Rightarrow \beta$. We describe an effective procedure producing all basic sequents provable in NIMELL(Φ). Let S_0 consists of all $T^!$-sequents from Φ, all $T^!$-sequents of the form (Id), and all $T^!$-sequents of the form:*

$$\alpha, (\alpha \to \beta) \Rightarrow \beta \quad (\beta \leftarrow \alpha), \beta \Rightarrow \alpha \quad \alpha, \beta \Rightarrow \alpha \otimes \beta$$

Assume S_n has already been defined. S_{n+1} is S_n enriched with all sequents arising by the following rules:

(1) *If $(!\alpha, !\alpha \Rightarrow \beta) \in S_n$ and $!\alpha \in T^!$, then $(!\alpha \Rightarrow \beta) \in S_{n+1}$;*
(2) *If $\alpha \Rightarrow \beta \in S_n$ and $!\gamma \in T^!$, then $\alpha, !\gamma \Rightarrow \beta \in S_{n+1}$;*
(3) *If $\alpha \Rightarrow \beta \in S_n$, then $!\alpha \Rightarrow \beta \in S_{n+1}$;*
(4) *If $!\alpha \Rightarrow \beta \in S_n$, then $!\alpha \Rightarrow !\beta \in S_{n+1}$;*
(5) *If $(\alpha, \beta \Rightarrow \gamma) \in S_n$ and $\alpha \otimes \beta \in T^!$, then $(\alpha \otimes \beta \Rightarrow \gamma) \in S_{n+1}$;*
(6) *If $(\alpha, \beta \Rightarrow \gamma) \in S_n$ and $(\alpha \to \gamma) \in T^!$, then $(\beta \Rightarrow \alpha \to \gamma) \in S_{n+1}$;*
(7) *If $(\alpha, \beta \Rightarrow \gamma) \in S_n$ and $(\gamma \leftarrow \beta) \in T^!$, then $(\alpha \Rightarrow \gamma \leftarrow \beta) \in S_{n+1}$;*
(8) *If $(\alpha \Rightarrow \beta) \in S_n$ and $(\sigma, \beta \Rightarrow \gamma) \in S_n$, then $(\sigma, \alpha \Rightarrow \gamma) \in S_{n+1}$;*
(9) *If $(\alpha \Rightarrow \beta) \in S_n$ and $(\beta, \sigma \Rightarrow \gamma) \in S_n$, then $(\alpha, \sigma \Rightarrow \gamma) \in S_{n+1}$;*
(10) *If $(\Gamma \Rightarrow \beta) \in S_n$ and $(\beta \Rightarrow \gamma) \in S_n$, then $(\Gamma \Rightarrow \gamma) \in S_{n+1}$.*

Obviously, $S_n \subseteq S_{n+1}$ for all $n \geq 0$. It is also noted that S_n is a finite set of basic sequents. Let S^T be the union of all S_n. The rules (1)-(7) are (Con$_!$), (Weak$_!$), (!L), (!R), ($\otimes$L), ($\to$R) and ($\leftarrow$R) restricted to basic sequents, and (8)-(10) describe the closure of basic sequents under (Cut) rule.

Lemma 7. *S^T can be constructed in polynomial time.*

Lemma 8. *Let $S(T)$ be a calculus which contains all the sequents from S^T but with the only rule of inference (Cut), then every basic sequent provable in $S(T)$ belongs to S^T.*

Lemma 9 ($S(T)$ Interpolation)**.** *If $\vdash_{S(T)} \Gamma[\Delta] \Rightarrow \alpha$, then there exists a $\gamma \in T^!$ such that $\vdash_{S(T)} \Delta \Rightarrow \gamma$ and $\vdash_{S(T)} \Gamma[\gamma] \Rightarrow \alpha$.*

By $\Gamma \Rightarrow_{T^!} \alpha$, we mean $\Gamma \Rightarrow \alpha$ has a proof in NIMELL(Φ) consisting of only $T^!$-sequents for any $T^!$-sequent $\Gamma \Rightarrow \alpha$.

Lemma 10. *For any $T^!$-sequent $\Gamma \Rightarrow \alpha$, $\Gamma \Rightarrow_{T^!} \alpha$ if and only if $\vdash_{S(T)} \Gamma \Rightarrow \alpha$.*

Theorem 11 (P-time Decidable)**.** *If Φ is finite, then NIMELL(Φ) is decidable in polynomial time.*

Further, one can prove the interpolation lemma for the distributive lattice extension of NIMELL and therefore has decidability. Accordingly, the sequent system for NDILL is obtained by adding axiom (D) $\alpha \& (\beta \oplus \gamma) \Leftrightarrow (\alpha \& \beta) \oplus (\alpha \& \gamma)$ and lattice rules:

$$\frac{\Gamma[\alpha_i] \Rightarrow \beta}{\Gamma[\alpha_1 \& \alpha_2] \Rightarrow \beta}(\&\mathrm{L}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \& \beta}(\&\mathrm{R}) \quad \frac{\Gamma[\alpha_1] \Rightarrow \beta \quad \Gamma[\alpha_2] \Rightarrow \beta}{\Gamma[\alpha_1 \oplus \alpha_2] \Rightarrow \beta}(\oplus\mathrm{L}) \quad \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \oplus \alpha_2}(\oplus\mathrm{R})$$

where the subscript i in (&L) and ($\oplus$R) equals to 1 or 2.

Remark 12. *E. Blaisdell et al [23] studied a system (called $\mathtt{acLL}_\Sigma$) that is relevant to NDILL and prove undecidability results for its fragments. The difference between these two systems is that NDILL owns the distributive lattice rule but another one only owns the pure lattice rule, which is the key to the decidability of the former while the undecidability of the latter.*

Given a sequent like $\alpha \Rightarrow \beta$, we define a set F to be all subformulas of formulas of α and β. We further define a set $!F$: if $\alpha \in F$, then $!\alpha \in !F$; If $!\alpha, !\beta \in F$, then $!\alpha \otimes !\beta \in !F$. Obviously, F and $!F$ are finite. Let T be the union of F and $!F$. Let $\overline{T}$ be the set T closed on the operations & and $\oplus$. Clearly, $\overline{T}$ is finite up to equivalence.

In what follows, we will prove interpolation lemmas for NDILL(Φ) and NIMELL(Φ). Such lemmas first came from W. Buszkowski [24] and later be discussed by Z. Lin [20].

Lemma 13 (NDILL(Φ) Interpolation)**.** *If $\vdash_{NDILL(\Phi)} \Gamma[\Delta] \Rightarrow \alpha$, then there is a $\gamma \in \overline{T}$ such that $\vdash_{NDILL(\Phi)} \Delta \Rightarrow \gamma$ and $\vdash_{NDILL(\Phi)} \Gamma[\gamma] \Rightarrow \alpha$ where T be the set of all subformulas of formulas of $\Gamma[\Delta] \Rightarrow \alpha$.*

Corollary 14 (NIMELL(Φ) Interpolation)**.** *If $\vdash_{NIMELL(\Phi)} \Gamma[\Delta] \Rightarrow \alpha$, then there is a $\gamma \in T$ such that $\vdash_{NIMELL(\Phi)} \Delta \Rightarrow \gamma$ and $\vdash_{NIMELL(\Phi)} \Gamma[\gamma] \Rightarrow \alpha$ where T be the set of all subformulas of formulas of $\Gamma[\Delta] \Rightarrow \alpha$.*

Theorem 15 (NDILL(Φ) decidable)**.** *If Φ is finite, then NDILL(Φ) is decidable.*

References

[1] J. Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1), 1-101, 1987.

[2] D. Miller. A logical analysis of modules in logic programming. *The Journal of Logic Programming*, 6(1-2), 79–108, 1989.

[3] J. S. Hodas and D. Miller. Logic programming in a fragment of intuitionistic linear logic. *Information and Computation*, 110(2), 327–365, 1994.

[4] D. Miller. A survey of linear logic programming. *Computational Logic: The Newsletter of the European Network of Excellence in Computational Logic*, 2(2), 63–67, 1995.

[5] C. Casadio. Non-commutative linear logic in linguistics. *Grammars*, 4(3), 167–185, 2001.

[6] G. Morrill. Categorial grammar: Logical syntax, semantics, and processing. *Oxford University Press*, 2010.

[7] Z. Lin. Full nonassociative Lambek calculus with modalities and its applications in type grammars. *Adam Mickiewicz University*, 2014.

[8] P. Lincoln, J. Mitchell, A. Scedrov and N. Shankar. Decision problems for propositional linear logic. *Annals of Pure and Applied Logic*, 56(1-3), 239–311, 1992.

[9] Y. Lafont. The finite model property for various fragments of linear logic. *The Journal of Symbolic Logic*, 62(4), 1202–1208, 1997.

[10] A. P. Kopylov. Decidability of linear affine logic. *Information and Computation*, 164(1), 173–198, 2001.

[11] C. A. Petri. Kommunikation mit automaten. *Universität at Bonn*, 1962.

[12] V. Mogbil. Phase semantics, proof nets and some decision problems in linear logic. *HAL*, 2001.

[13] C Reutenauer. Aspects mathématiques des réseaux de Petri. *Elsevier Masson*, 1989.

[14] P. De. Groote, B. Guillaume, and S. Salvati. Vector addition tree automata. *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*, 64-73, 2004.

[15] R. Lazić and S. Schmitz. Nonelementary complexities for branching VASS, MELL, and extensions. *ACM Transactions on Computational Logic (TOCL)*, 16(3), 1–30, 2015.

[16] L. Straßburger. System NEL is undecidable. *Electronic Notes in Theoretical Computer Science*, 84, 166–177, 2003.

[17] K. Bimbó. The finite model property for various fragments of linear logic. *Theoretical Computer Science*, 597, 1–17, 2015.

[18] L. Straßburger. On the decision problem for MELL. *Theoretical Computer Science*, 768, 91–98 2019.

[19] H. Tanaka. A note on undecidability of propositional non-associative linear logics. *arXiv preprint arXiv: 1909.13444*, 2019.

[20] Z. Lin. Non-associative Lambek calculus with modalities: interpolation, complexity and FEP. *Logic Journal of the IGPL*, 22(3), 494-512, 2014.

[21] Z. Lin. Modal Nonassociative Lambek Calculus with Assumptions: Complexity and Context-Freeness. *LATA*, 414-425, 2010.

[22] W. Buszkowski. Lambek calculus with nonlogical axioms. *Language and Grammar. Studies in Mathematical Linguistics and Natural Language*, 77-93, 2005.

[23] E. Blaisdell, M. Kanovich, S.L. Kuznetsov, E. Pimentel, and A. Scedrov. Non-associative, Non-commutative Multi-modal Linear Logic. *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, 13385, 449–467, 2022.

[24] W. Buszkowski. Interpolation and FEP for logics of residuated algebras. *Logic Journal of IGPL*, 19(3), 437–454, 2011.

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INVOLUTIVE COMMUTATIVE RESIDUATED LATTICE WITHOUT UNIT: LOGICS AND DECIDABILITY

YIHENG WANG AND ZHE LIN

The residuated lattice-ordered monoid, or residuated lattice for short, was first introduced by Ward, et al. [1, 2, 3] in the early 20th century. A residuated lattice is a structure $\mathsf{F} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, \backslash, /, 1)$ is a residuated monoid where 1 is the unit for fusion "$\cdot$", and $\backslash, \cdot, /$ are binary operations satisfying the following law of residuation:

$$\text{(res)} \quad a \cdot b \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b.$$

The class of all residuated lattices will be denoted by $\mathcal{RL}$. The study of $\mathcal{RL}$ can be traced back to the work of W. Krull [1] and that of Morgan Ward and R. P. Dilworth [2, 3]. The theory has been studied in several branches, including lattice-ordered groups, substructural logics, and mathematical linguistics. The class of all residuated semigroups is denoted by $\mathcal{RSG}$. $\mathcal{RL}$ without the unit 1 denoted by $\mathcal{RL}^-$ forms the residuated lattice-ordered semigroup. $\mathcal{RSG}$, $\mathcal{RL}^-$, and their non-associative variants are widely used as the algebraic base of categorial grammars for natural language processing (cf. [4, 5, 6]). The commutative $\mathcal{RL}$ ($\mathcal{RL}^-$), denoted by $c\mathcal{RL}$ ($c\mathcal{RL}^-$) is $\mathcal{RL}$ ($\mathcal{RL}^-$) additionally satisfying the commutativity for "$\cdot$" i.e. $a \cdot b = b \cdot a$. Consequently in $c\mathcal{RL}$ ($c\mathcal{RL}^-$), $a\backslash b = b/a$.

There are lots of literature on $\mathcal{RL}$ enriched with an involutive negation (in) denoted by $In\mathcal{RL}$ (cf. [7, 8]). An involutive negation over $\mathcal{RL}$ and $\mathcal{RL}^-$ is a unary operation satisfying the following two conditions: (in) $a\backslash\neg b = \neg a/b$ and (dn) $\neg\neg a = a$ (cf. [7]). Let $0 = \neg 1$. Then $a\backslash 0 = \neg a/1 = \neg a$. Thus $\neg a = a\backslash 0 = 0/a$. Therefore the contraposition (ctr) follows: if $a \leq b$, then $\neg b \leq \neg a$. In $c\mathcal{RL}$, (in) is equivalent to (cin): $a\backslash\neg b = b\backslash\neg a$. If one replaces (dn) by $a \leq \neg\neg a$, then one obtains the quasi-involutive negation. Hereafter we denote $Inc\mathcal{RL}$ and $Inc\mathcal{RL}^-$ for involutive $c\mathcal{RL}$ and $c\mathcal{RL}^-$ respectively. Evidently, quasi-involutive negation and involutive negation are minimal negation and De Morgan negation respectively. De Morgan negation is a unary operation satisfying (ctr) and (dn) (cf. [9]). If one omits $\neg\neg a \leq a$ from a De Morgan negation, then one obtains the minimal negation. The concept of minimal negation was introduced by Kolmogoroff [10], but it is systematically studied by Johansson [11]. De Morgan algebras (also called "quasi-Boolean algebras"), which are (not necessarily bounded) distributive lattices with a De Morgan negation, were introduced by Bialynicki-Birula and Rasiowa [12]. This type of algebra was also investigated by Moisil [13] under the term De Morgan lattices, and by Kalman [14] under the term distributive i-lattices.

Substructural logics are the logics whose algebraic models are residuated structures. The research of substructural logics and residuated lattices has been risen by Ono and other people (cf. [6, 15, 16, 17]) over the last four decades. The logic of $\mathcal{RL}$ under the name full Lambek calculus (FL), is a lattice extension of Lambek calculus (L) with the unit. Negative extensions of substructural logics corresponding to $\mathcal{RL}$ and its variants are considered in different literatures. $Inc\mathcal{RL}$ (also named FL_e-monoids) are studied by Jenei and Ono [8]. Galatos and Jipsen [18] show that $In\mathcal{RL}$ and its nonassociative variants have the finite model property. Cut-free sequent calculi for the corresponding algebras are presented in the same paper. Buszkowski [19] considers involutive residuated groupoids. The logic of these algebras is called classical nonassociative Lambek Calculus (CNL). Cut-free one side sequent calculus is presented and polynomial decidability is proved for CNL and CNL with the unit (CNL1).

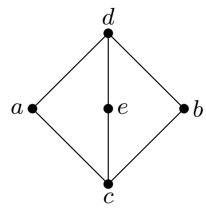
We continue this line of research. We consider (quasi) involutive negations over $c\mathcal{RL}^-$. $c\mathcal{RL}^-$ and $c\mathcal{RL}$ are essentially different. For instance, the inequation $a/(b/b) \leq a$ holds in $c\mathcal{RL}$ but not in $c\mathcal{RL}^-$. Moreover $a \cdot \neg a \leq 0$ is a basic law in $Inc\mathcal{RL}$. However, this expression lacks sense in $Inc\mathcal{RL}^-$ without 0. Further up to the knowledge of the authors, the algebraic properties of $Inc\mathcal{RL}^-$ are not well investigated. And the logical characterization, cut-free sequent calculus, and the decidability results of the corresponding logic remain open. Clearly, these results cannot be easily adapted from the ones for $Inc\mathcal{RL}$. In the present paper, we study $Inc\mathcal{RL}^-$ and its corresponding logic InFL_e^-. A sequent calculus for InFL_e^- is presented and the decidability results are established. Our method is inspired by Lin et al. [20].

Definition 1. *An algebra $\mathsf{F} = (A, \wedge, \vee, \cdot, \backslash, \neg)$ is a $Inc\mathcal{RL}^-$ if and only if:*

- $(A, \wedge, \vee, \cdot, \backslash)$ *is a* $c\mathcal{RL}^-$*;*
- $\neg$ *is an operation on A satisfying the following properties:*
  (dn) $\neg\neg a = a$;
  (in) $a\backslash\neg b = b\backslash\neg a$.

Lemma 2. *In $Inc\mathcal{RL}$, one has $\neg a = a\backslash 0$ and $a \cdot \neg a = 0$ where $0 = \neg 1$, which doesn't fit $Inc\mathcal{RL}^-$.*

We have the following Figure 5 example shows that $Inc\mathcal{RL}^-$ is essentially different with $Inc\mathcal{RL}$:



$\cdot$	a	b	c	d	e
a	b	a	c	d	d
b	a	b	c	d	d
c	c	c	c	c	c
d	d	d	c	d	d
e	d	d	c	d	d

	a	b	c	d	e
$\neg$	a	b	d	c	e

$\backslash$	a	b	c	d	e
a	b	a	c	d	c
b	a	b	c	d	c
c	d	d	d	d	d
d	c	c	c	d	c
e	c	c	c	d	c

FIGURE 5. An example of $Inc\mathcal{RL}^-$

Definition 3. *The set of formulas (terms) $\mathcal{F}$ for InFL_e^- is defined inductively as follows:*

$$\mathcal{F} \ni \alpha ::= p \mid \alpha \cdot \beta \mid \alpha \backslash \beta \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \neg \alpha$$

*where $p \in$ **Var**.*

Definition 4. The Gentzen-style sequent calculus G for InFL_e^-:

(1) Axioms:

$$(\mathrm{Id})\alpha \Rightarrow \alpha \quad (\mathrm{DN2})\neg\neg\alpha \Rightarrow \alpha$$

(2) Logical rules:

$$\frac{\alpha \cdot \beta \Rightarrow \gamma}{\beta \Rightarrow \alpha \backslash \gamma}(\mathrm{RES}\backslash) \quad \frac{\beta \Rightarrow \alpha \backslash \gamma}{\alpha \cdot \beta \Rightarrow \gamma}(\mathrm{RES}^-\backslash)$$

$$\frac{\alpha \Rightarrow \beta}{\alpha \wedge \gamma \Rightarrow \beta}(\wedge \mathrm{L}) \quad \frac{\alpha \Rightarrow \beta \quad \alpha \Rightarrow \gamma}{\alpha \Rightarrow \beta \wedge \gamma}(\wedge \mathrm{R})$$

$$\frac{\alpha \Rightarrow \beta \quad \gamma \Rightarrow \beta}{\alpha \vee \gamma \Rightarrow \beta}(\vee \mathrm{L}) \quad \frac{\alpha \Rightarrow \beta}{\alpha \Rightarrow \beta \vee \gamma}(\vee \mathrm{R})$$

$$\frac{\alpha \cdot \beta \Rightarrow \neg\gamma}{\gamma \cdot \beta \Rightarrow \neg\alpha}(\neg)$$

(2) Cut rule:

$$\frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}(\mathrm{Cut})$$

Theorem 5 (Soundness and Completeness)**.** G *is sound and complete with respect to $Inc\mathcal{RL}^-$s.*

The sequent calculus for $c\mathcal{RL}^-$ with a quasi-involutive negation (denoted by $qInc\mathcal{RL}^-$) is just G omitting $\neg\neg\alpha \Rightarrow \alpha$, denoted by qG.

Definition 6 (Kolmogorov Translation)**.** *The Kolmogorov translation is a function $ko : \mathcal{F} \to \mathcal{F}$ defined inductively as follows:*

(1) $ko(\alpha) = \neg\neg\alpha$ *if α is atomic;*
(2) $ko(\alpha \wedge \beta) = \neg\neg(ko(\alpha) \wedge ko(\beta))$;
(3) $ko(\alpha \vee \beta) = \neg\neg(ko(\alpha) \vee ko(\beta))$;
(4) $ko(\neg\alpha) = \neg ko(\alpha)$;
(5) $ko(\alpha \cdot \beta) = \neg\neg(ko(\alpha) \cdot ko(\beta))$;
(6) $ko(\alpha \backslash \beta) = \neg\neg(ko(\alpha) \backslash ko(\beta))$.

Lemma 7. $\vdash_{\mathsf{qG}} ko(\alpha) \Rightarrow ko(\beta)$ *iff* $\vdash_{\mathsf{G}} \alpha \Rightarrow \beta$ *where $ko()$ is the Kolmogorov translation.*

Theorem 8. G *can be embedded into* qG.

Corollary 9. $Inc\mathcal{RL}^-$ *can be embedded into* $qInc\mathcal{RL}^-$.

Definition 10. *An algebra $\ddot{\mathsf{F}} = (A, \wedge, \vee, \cdot, \backslash, *, \to, \bot, \top)$ is a bounded commutative bi-residuated lattice-ordered semigroup (denoted by $cb\mathcal{RL}^-$) such that $(A, \wedge, \vee, \cdot, \backslash)$ is a $c\mathcal{RL}^-$, and $(A, \wedge, \vee, *, \to)$ is a commutative residuated lattice-ordered groupoid. $\bot, \top \in A$ are the least and greatest elements in A. Two fusions $*, \cdot$ are operations on A satisfying the following condition for all $a, b, c \in A$:*

$$\textit{If } (a \cdot b) * c = \bot, \textit{ then } (a \cdot c) * b = \bot.$$

One defines $\neg a = a \to \bot$, then one has $a * \neg a \leq \bot$.

Theorem 11. *Every $qInc\mathcal{RL}^-$ can be extended to a $cb\mathcal{RL}^-$.*

Definition 12. *The set of formulas (terms) $\mathcal{F}$ for the logic of $cb\mathcal{RL}^-$ (denoted by bFL_e^-) is defined inductively as follows:*

$$\mathcal{F} \ni \alpha ::= p \mid \alpha \cdot \beta \mid \alpha \backslash \beta \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha * \beta \mid \alpha \to \beta \mid \bot$$

where $p \in$ **Var**. *We use the abbreviations $\neg\alpha := \alpha \to \bot$ and $\top = \neg\bot$.*

Definition 13. *Let , and ; be structural counterparts for $\cdot$ and $*$ respectively. The set of all formula structures $\mathcal{FS}$ is defined inductively as follows:*

$$\mathcal{FS} \ni \Gamma ::= \alpha \mid \Gamma, \Gamma \mid \Gamma; \Gamma$$

Definition 14. The Gentzen-style sequent calculus Gb for bFL_e^- can be obtained from G by deleting the (DN2) and $(\neg)$ but adding the following rules:

(1) Logical rules:

$$\frac{\Gamma[\alpha,\beta] \Rightarrow \gamma}{\Gamma[\alpha \cdot \beta] \Rightarrow \gamma}(\cdot\mathrm{L}) \quad \frac{\Gamma_1 \Rightarrow \alpha \quad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta}(\cdot\mathrm{R}) \quad \frac{\Gamma[\alpha;\beta] \Rightarrow \gamma}{\Gamma[\alpha * \beta] \Rightarrow \gamma}(*\mathrm{L}) \quad \frac{\Gamma_1 \Rightarrow \alpha \quad \Gamma_2 \Rightarrow \beta}{\Gamma_1; \Gamma_2 \Rightarrow \alpha * \beta}(*\mathrm{R})$$

$$\frac{\Delta \Rightarrow \alpha \quad \Gamma[\beta] \Rightarrow \gamma}{\Gamma[\Delta; \alpha \to \beta] \Rightarrow \gamma}(\to \mathrm{L}) \quad \frac{\alpha; \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}(\to \mathrm{R}) \quad \frac{\Delta \Rightarrow \bot}{\Gamma[\Delta] \Rightarrow \alpha}(\bot)$$

(2) Structural rules:

$$\frac{(\Delta_1, \Delta_2); \Delta_3 \Rightarrow \bot}{(\Delta_1, \Delta_3); \Delta_2 \Rightarrow \bot}(\mathrm{R} - \bot)$$

$$\frac{\Gamma[\Delta_1, \Delta_2] \Rightarrow \beta}{\Gamma[\Delta_2, \Delta_1] \Rightarrow \beta}(\mathrm{Ex}) \quad \frac{\Gamma[\Delta_1; \Delta_2] \Rightarrow \beta}{\Gamma[\Delta_2; \Delta_1] \Rightarrow \beta}(\mathrm{Ex}^{;}) \quad \frac{\Gamma[\Delta_1, (\Delta_2, \Delta_3)] \Rightarrow \beta}{\Gamma[(\Delta_1, \Delta_2), \Delta_3] \Rightarrow \beta}(\mathrm{As}_1) \quad \frac{\Gamma[(\Delta_1, \Delta_2), \Delta_3] \Rightarrow \beta}{\Gamma[\Delta_1, (\Delta_2, \Delta_3)] \Rightarrow \beta}(\mathrm{As}_2)$$

Theorem 15 (Soundness and Completeness)**.** Gb *is sound and complete with respect to $cb\mathcal{RL}^-$s.*

Theorem 16 (Cut-Elimination)**.** $\vdash_{\mathsf{Gb}} \Gamma \Rightarrow \beta$ *iff* $\vdash_{\mathsf{Gb}} \Gamma \Rightarrow \beta$ *without any application of* (Cut) *rule.*

Corollary 17 (Subformula Property)**.** *If* $\vdash_{\mathsf{Gb}} \Gamma \Rightarrow \alpha$*, then there exists a derivation of* $\Gamma \Rightarrow \alpha$ *in* Gb *such that all formulas appearing in the proof are subformulas of formulae appearing in* $\Gamma \Rightarrow \alpha$*.*

Theorem 18 (Decidability)**.** Gb *and bFL_e^- are decidable.*

Theorem 19 (Conservative Extension)**.** *For any* qG *sequent* $\alpha \Rightarrow \beta$*,* $\vdash_{\mathsf{qG}} \alpha \Rightarrow \beta$ *iff* $\vdash_{\mathsf{Gb}} \alpha \Rightarrow \beta$*.*

Theorem 20 (Decidability)**.** qG *is decidable.*

Corollary 21. G *and $InFL_e^-$ are decidable.*

References

[1] W. Krull, "Axiomatische Begrundung der allgemeinen Idealtheorie," *Sitzungsberichte der physikalisch medizinischen Societat der Erlangen*, vol. 56, pp. 47–63, 1924.
[2] M. Ward, "Residuation in structures over which a multiplication is defined," *Duke Math. J.*, 1937, [CrossRef].
[3] R. P. Dilworth, "Abstract residuation over lattices," *Bulletin of the American Mathematical Society*, vol. 44, pp. 262–268, 1938, [CrossRef].
[4] J. Lambek, "The mathematics of sentence structure," *The American Mathematical Monthly*, vol. 65, no. 3, pp. 154–170, 1958, [CrossRef].
[5] M. Kanazawa, "The Lambek calculus enriched with additional connectives," *Journal of Logic, Language and Information*, vol. 1, pp. 141–171, 1992, [CrossRef].
[6] W. Buszkowski, "Lambek calculus and substructural logics," *Linguistic Analysis*, vol. 36, no. 1, p. 15, 2006.
[7] N. Galatos and J. G. Raftery, "Adding involution to residuated structures," *Studia Logica*, vol. 77, pp. 181–207, 2004, [CrossRef].
[8] S. Jenei and H. Ono, "On involutive fl e-monoids," *Archive for Mathematical Logic*, vol. 51, pp. 719–738, 2012, [CrossRef].
[9] J. M. Dunn, "A comparative study of various model-theoretic treatments of negation: a history of formal negation," *What is negation?*, pp. 23–51, 1999, [CrossRef].
[10] A. Kolmogoroff, "Zur Deutung der intuitionistischen Logik," *Mathematische Zeitschrift*, vol. 35, pp. 58–65, 1932, [CrossRef].
[11] I. Johansson, "Der Minimalkalkiil, ein reduzierter intuitionistischer Formalismus," *CompositioMathematica*, vol. 4, p. 119–136, 1936.
[12] A. Bialynicki-Birula and H. Rasiowa, "On the representation of quasi-boolean algebras," *Bulletin of the Polish Academy of Sciences*, vol. Cl. III 5, no. 5, pp. 259–261, 1957, [CrossRef].
[13] G. C. Moisil, "Recherches sur l'algèbre de la logique," *Ann. Sci. Univ. Jassy*, vol. 22, pp. 1–117, 1935.
[14] J. A. Kalman, "Lattices with involution," *Transactions of the American Mathematical Society*, vol. 87, no. 2, pp. 485–491, 1958, [CrossRef].
[15] P. J. Schroeder-Heister and K. Došen, *Substructural logics*. Oxford, England: Oxford University Press, 1993.
[16] G. Restall, *An introduction to substructural logics*. London, England: Routledge, 2002, [CrossRef].
[17] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated lattices: an algebraic glimpse at substructural logics*. Amsterdam, Netherlands: Elsevier, 2007, [CrossRef].

[18] N. Galatos and P. Jipsen, “Residuated frames with applications to decidability,” *Transactions of the American Mathematical Society*, vol. 365, no. 3, pp. 1219–1249, 2013, [CrossRef].
[19] W. Buszkowski, “On classical nonassociative Lambek calculus,” in *Logical Aspects of Computational Linguistics. Celebrating 20 Years of LACL (1996–2016) 9th International Conference, LACL 2016, Nancy, France, December 5-7, 2016, Proceedings.* Springer, 2016, pp. 68–84, [CrossRef].
[20] Z. Lin, M. K. Chakraborty, and M. Ma, “Residuated algebraic structures in the vicinity of pre-rough algebra and decidability,” *Fundamenta Informaticae*, vol. 179, no. 3, pp. 239–274, 2021, [CrossRef].

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